Robot Arm Process. Modeling and Control

6.1 Euler-Lagrange equations

The Lagrangian is defined as

L = K - P

where K represents the total kinetic energy of the system and P represents the total potential energy of the system.

The Euler-Lagrange equations that describe the dynamics of a $n - DOF$ mechanical system are [1] :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i, \qquad i = 1, ..., n, \tag{6.1}$$

where q_i represent generalized coordinates (in our case the joint angles) and τ_i generalized forces (in our case motor torques) [2] .

The matrix form of the Euler-Lagrange equations is:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \tag{6.2}$$

where $q = [q_1, ..., q_n]^T, \tau = [\tau_1, ..., \tau_n]^T$.

The matrix $D(q)$ is called inertia matrix, it is symmetric and positive definite, and can be expressed in terms of the kinetic energy:

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\sum_{i,j}^{n} d_{i,j}(q)\dot{q}_i\dot{q}_j. \tag{6.3}$$

[1] The Euler-Lagrange equations are also used in optimal control and calculus of variations. See [3] for an interesting discussion on the interplay between the physical interpretation and the mathematical insight.

[2] This presentation is based on [1]. For a formal derivation of the Euler-Lagrange equations from Newton's Laws based on the principle of virtual work see chapter 6.1 of the book.

The matrix $C(q)$ takes into account centrifugal and Coriolis terms, and each $k, j-th$ matrix element can be calculated as:

$$c_{kj} = \frac{1}{2}\sum_{i=1}^{n}\underbrace{\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}}_{c_{ijk}}\dot{q}_i. \tag{6.4}$$

The last term $G(q)$, sometimes called gravity term, is a column vector $G = [g_1 ... g_n]^T$, where each $k - th$ term is derived from the potential energy:

$$g_k(q) = \frac{\partial P}{\partial q_k}, \qquad k = 1, ..., n. \tag{6.5}$$

6.2 A 2DOF robot arm with spatial movement

Consider a 2DOF robot arm with two revolute joints, that can move in a 3D Cartesian space, with the schematic representation from Figure 6.1. Because the first rotation axis is on the X axis, an the second on the Y axis, that robot can move in a 3D space.

6.2.1 Geometric Model

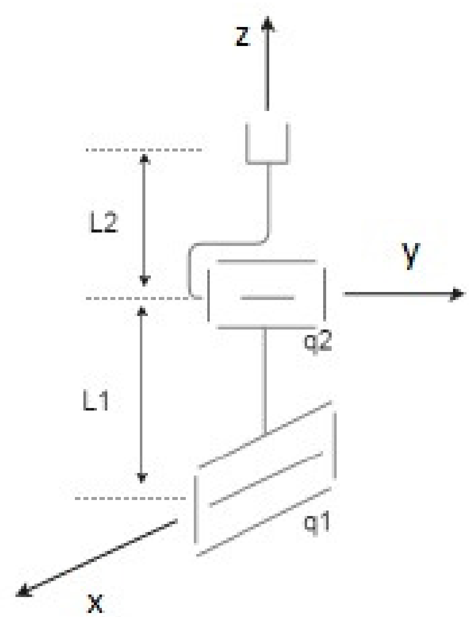

Figure 6.1: Schematic representation of a 2DOF robot arm

The geometric model can be derived through transformation matrices from the base frame to the end effector frame. The base frame coincides with the first frame (that is the frame of joint 1, with origin O_1 in the center of the joint). Thus the transformation matrix T_{01} is simply a rotation around X:

$$T_{01} = Rot(x, q_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(q_1) & -sin(q_1) & 0 \\ 0 & sin(q_1) & cos(q_1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From Frame 1 we arrive at Frame 2 (corresponding to the joint 2) through a translation on Z and a rotation around Y (T_{12}):

$$T_{12} = Transl(z, L_1) \cdot Rot(y, q_2) = \begin{bmatrix} cos(q2) & 0 & sin(q2) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(q2) & 0 & cos(q2) & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, the end effector frame is obtained through a translation on Z (T_{23}):

$$T_{23} = Transl(z, L_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix from the based frame to the end effector, that is the geometric model is obtained through multiplication:

$$T = T_{03} = T_{01} \cdot T_{12} \cdot T_{23} = \begin{bmatrix} cos(q_2) & 0 & sin(q_2) & L_2 sin(q_2) \\ sin(q_1)sin(q_2) & cos(q_1) & -cos(q_2)sin(q_1) & -sin(q_1)(L_1 + L_2 cos(q_2)) \\ -cos(q_1)sin(q_2) & sin(q_1) & cos(q_1)cos(q_2) & cos(q_1)(L_1 + L_2 cos(q_2)) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.6}$$

The position of the end effector with respect to the joint angles q_1 and q_2 is given by the first three elements of the 4th column:

$$x = L_2 sin(q_1), y = L_1 sin(q_1) - L_2 sin(q_1) cos(q_2), z = L_1 cos(q_1) + L_2 cos(q_1) cos(q_2). \tag{6.7}$$

The orientation of the end effector is given by the submatrix R (lines 1-3 and columns 1-3 of T):

$$R = \begin{bmatrix} cos(q_2) & 0 & sin(q_2) \\ sin(q_1)sin(q_2) & cos(q_1) & -cos(q_2)sin(q_1) \\ -cos(q_1)sin(q_2) & sin(q_1) & cos(q_1)cos(q_2) \end{bmatrix}. \tag{6.8}$$

6.2.2 Jacobian

The Jacobian relates the joint velocities to the linear and angular velocities of the end effector [3]. For the 2DOF robot arm from Figure 6.1, with the geometric model 6.6, the Jacobian is:

$$J = \begin{bmatrix} 0 & \frac{L_2}{2}cos(q_2) \\ -\frac{L_2}{2}cos(q_1)cos(q_2) - L_1cos(q_1) & \frac{L_2}{2}sin(q_1)sin(q_2) \\ -\frac{L_2}{2}sin(q_1)cos(q_2) - L_1sin(q_1) & -\frac{L_2}{2}cos(q_1)sin(q_2) \\ 1 & 0 \\ 0 & cos(q_1) \\ 0 & sin(q_1) \end{bmatrix}. \tag{6.9}$$

Thus, if we refer to link 2, the angular and linear Jacobians are:

$$J_{vc2} = J_{vc} = \begin{bmatrix} 0 & \frac{L_2}{2}cos(q_2) \\ -\frac{L_2}{2}cos(q_1)cos(q_2) - L_1cos(q_1) & \frac{L_2}{2}sin(q_1)sin(q_2) \\ -\frac{L_2}{2}sin(q_1)cos(q_2) - L_1sin(q_1) & -\frac{L_2}{2}cos(q_1)sin(q_2) \end{bmatrix} \tag{6.10}$$

$$J_{\omega 2} = J_{\omega} = \begin{bmatrix} 1 & 0 \\ 0 & cos(q_1) \\ 0 & sin(q_1) \end{bmatrix} \tag{6.11}$$

Further on, the angular and linear Jacobians for link 1 [4] can be determined as:

$$J_{vc1} = \begin{bmatrix} 0 & 0 \\ -\frac{L_1}{2}cos(q_1) & 0 \\ -\frac{L_1}{2}sin(q_1) & 0 \end{bmatrix}, \tag{6.12}$$

$$J_{\omega 1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{6.13}$$

[3] Check chapter 6 for more details on how to derive it.

[4] Reconsider the calculations as if link 2 does not exist.

6.2.3 Lagrangian

The Lagrangian is composed out of kinetic energy and potential energy. The kinetic energy has a translational and a rotational component

$$K = K_{transl} + K_{rot} \tag{6.14}$$

given by the expressions:

$$K_{transl} = \frac{1}{2} m_1 v_{c1}^T v_{c1} + \frac{1}{2} m_2 v_{c2}^T v_{c2} = \frac{1}{2} \dot{q}^T (m_1 J_{vc1}^T J_{vc1} + m_2 J_{vc2}^T J_{vc2}) \dot{q} \tag{6.15}$$

and

$$K_{rot} = \frac{1}{2} \dot{q}^T (J_{\omega 2}^T R_2 I_2 R_2^T J_{\omega 2} + J_{\omega 1}^T R_1 I_1 R_1^T J_{\omega 1}) \dot{q} \tag{6.16}$$

with[5]

$$R_2 = R, \quad R_1 = T_{01}(1:3, 1:3), I_2 = diag\{0, I_{2y}, 0\}, \quad I_1 = diag\{I_{1x}, 0, 0\}. \tag{6.17}$$

After calculating the expressions for both components of the kinetic energy, we obtain the inertia matrix $D(q)$ as

$$D(q) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} I_{1x} + \frac{L_1^2 m_1}{4} + L_1^2 m_2 + \frac{L_2^2 m_2}{4} cos^2(q_2) + L_1 L_2 m_2 cos(q_2) & 0 \\ 0 & \frac{m_2 L_2^2}{4} + I_{2y} \end{bmatrix} \tag{6.18}$$

In deriving matrix $C(q, \dot{q})$, we first calculate each c_{ijk} term from (6.4):

$$c_{111} = \frac{\partial d_{11}}{\partial q_1} + \frac{\partial d_{11}}{\partial q_1} - \frac{\partial d_{11}}{\partial q_1} = 0,$$

$$c_{112} = \frac{\partial d_{21}}{\partial q_1} + \frac{\partial d_{21}}{\partial q_1} - \frac{\partial d_{11}}{\partial q_2} = \frac{L_2^2 m_2}{4} sin(2q_2) + L_1 L_2 m_2 sin(q_2),$$

[5] Because in practice the off diagonals terms of the inertia matrices are neglijible, we consider here only the Principle Moments of Inertia corresponding to each rotation axis (I_x, I_y or I_z). Note that the inertia moments are expressed with respect to the body attached frame.

$$c_{121} = \frac{\partial d_{12}}{\partial q_1} + \frac{\partial d_{11}}{\partial q_2} - \frac{\partial d_{12}}{\partial q_1} = -\frac{L_2^2 m_2}{4} sin(2q_2) - L_1 L_2 m_2 sin(q_2),$$
$$c_{122} = \frac{\partial d_{22}}{\partial q_1} + \frac{\partial d_{21}}{\partial q_2} - \frac{\partial d_{12}}{\partial q_2} = 0,$$
$$c_{211} = \frac{\partial d_{11}}{\partial q_2} + \frac{\partial d_{12}}{\partial q_1} - \frac{\partial d_{21}}{\partial q_1} = c_{121},$$
$$c_{212} = \frac{\partial d_{21}}{\partial q_2} + \frac{\partial d_{22}}{\partial q_1} - \frac{\partial d_{21}}{\partial q_2} = 0,$$
$$c_{221} = \frac{\partial d_{12}}{\partial q_2} + \frac{\partial d_{12}}{\partial q_2} - \frac{\partial d_{22}}{\partial q_1} = 0,$$
$$c_{222} = \frac{\partial d_{22}}{\partial q_2} + \frac{\partial d_{22}}{\partial q_2} - \frac{\partial d_{22}}{\partial q_2} = 0.$$

In the end we obtain the matrix:

$$C(q,\dot{q}) = \begin{bmatrix} -\frac{L_2^2 m_2}{8} sin(2q_2)\dot{q}_2 - \frac{1}{2} L_1 L_2 m_2 sin(q_2)\dot{q}_2 & -\frac{L_2^2 m_2}{8} sin(2q_2)\dot{q}_1 - \frac{1}{2} L_1 L_2 m_2 sin(q_2)\dot{q}_1 \\ \frac{L_2^2 m_2}{8} sin(2q_2)\dot{q}_1 + \frac{1}{2} L_1 L_2 m_2 sin(q_2)\dot{q}_1 & 0 \end{bmatrix} \tag{6.19}$$

The potential energy is determined by multiplying the mass by the gravitational acceleration and the height at the center of mass:

$$P_1 = m_1 g \frac{L_1}{2} cos(q_1),\ P_2 = m_2 g \Big(L_1 cos(q_1) + \frac{L_2}{2} cos(q_1) cos(q_2) \Big),\ P = P_1 + P_2. \tag{6.20}$$

Based on (6.5), the gravity term is determined as:

$$G(q) = \begin{bmatrix} -\frac{m_1 g L_1 + 2 m_2 g L_1}{2} sin(q_1) - \frac{m_2 g L_2}{2} sin(q_1) cos(q_2) \\ -\frac{m_2 g L_2}{2} cos(q_1) sin(q_2) \end{bmatrix} \tag{6.21}$$

This completes the dynamic model for our robot arm.

6.3 Proposed problems

1. Consider a robotic structure with only 1 degree of freedom rotating around z axis, for which $l = 1\ m$ and $m = 1\ kg$.

a) Compute the Euler-Lagrange equations for the robot.

b) Implement in Matlab Simulink the robot model using a user-defined function and having the torque as a sine wave.

2. Consider the 2DOF robotic structure from Figure 6.1, for which $L1 = 0.095\ m$, $L2 = 0.1\ m$, $m1 = 0.095\ kg$, $m2 = 0.37\ kg$, and $I1 = I2 = 0.025\ kg/m^2$

a) Find the DGM using the D-H convention.

b) Using the provided C, D, and G matrices, implement in Matlab Simulink the robot model, having two sine waves as the input joint torques.

$$D = \begin{bmatrix} 0.003515c_2 + 0.000925c_2^2 + 0.026254 & 0 \\ 0 & 0.023625 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.24 - 0.0017575\dot{q}_2 s_2 - 0.0004625\dot{q}_2 sin(2 * q_2) & -0.0004625\dot{q}_1 sin(2 * q_2) - 0.0017575\dot{q}_1 * s_2 \\ 0.0004625\dot{q}_1 sin(2 * q_2) + 0.0017575\dot{q}_1 s_2 & 0.16 \end{bmatrix}$$

$$G = \begin{bmatrix} (-0.1815c_2 - 0.1376)s_1 \\ -0.18149c_1 s_2 \end{bmatrix}$$

$$\tau = \begin{bmatrix} sin(t) \\ sin(t) \end{bmatrix}$$