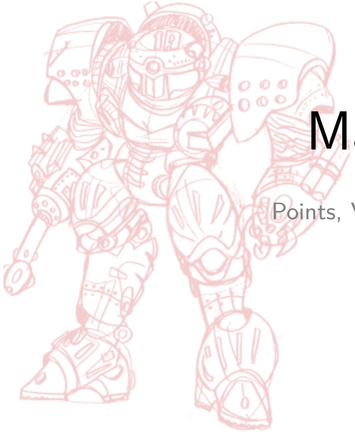


Mathematical background

Points, Vectors, Coordinate systems, Transformation matrices

TECHNICAL UNIVERSITY
OF CLUJ-NAPOCA
ROMANIA

Last update: September 30, 2024

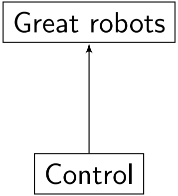
Grand scheme

The big picture

Great robots

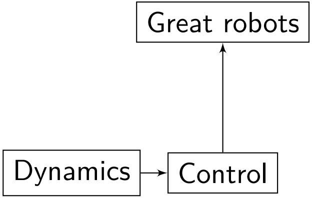
Grand scheme

The big picture



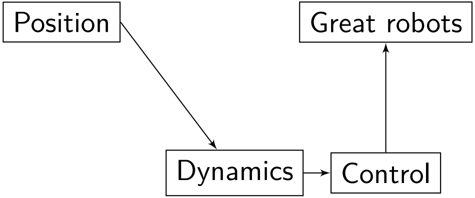
Grand scheme

The big picture



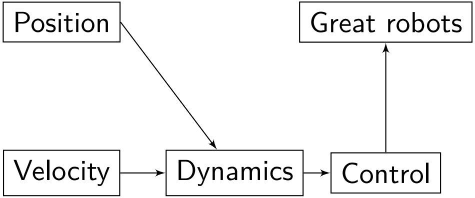
Grand scheme

The big picture



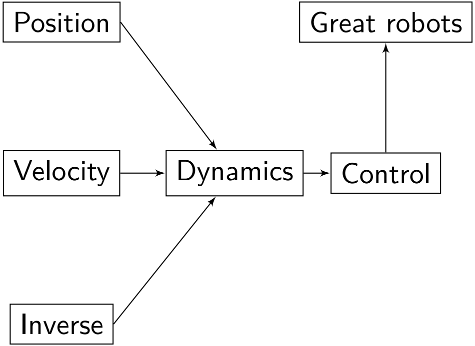
Grand scheme

The big picture



Grand scheme

The big picture



Grand scheme

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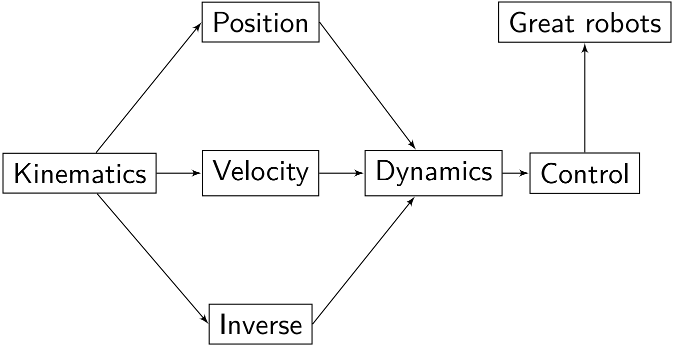
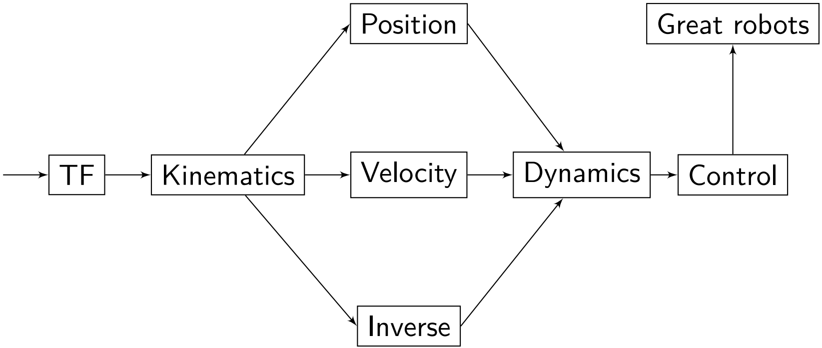


Grand scheme

The big picture



Agenda

- Definitions of coordinate systems
- Rule of the right hand
- Points and vectors in $\mathbb{R}^2$ & $\mathbb{R}^3$
- Transformation matrices
- Homogeneous transformations

Coordinate systems

Cartesian coordinates

In simple words

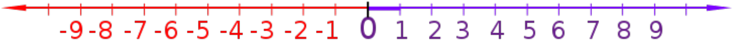
A coordinate system is a mathematical tool that allows us to describe the position of objects in space using numbers. Each coordinate system has axes, equal in number to the number of dimensions of space.

Properties

- The axes must be perpendicular to each other
- The length of the axes is one unit
- Each point has n number of coordinates, equal to the number of axes
- There can be more than one coordinate system to describe a certain space

Coordinate systems

The $\mathbb{R}^1$ case

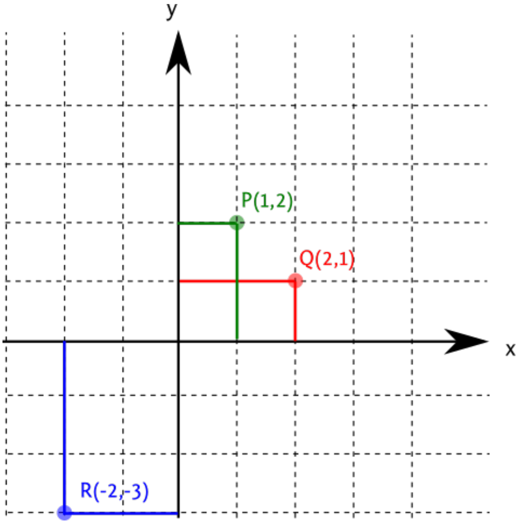


A point P in $\mathbb{R}^1$ space is represented as $P = [2]$

Coordinate systems

The $\mathbb{R}^2$ case

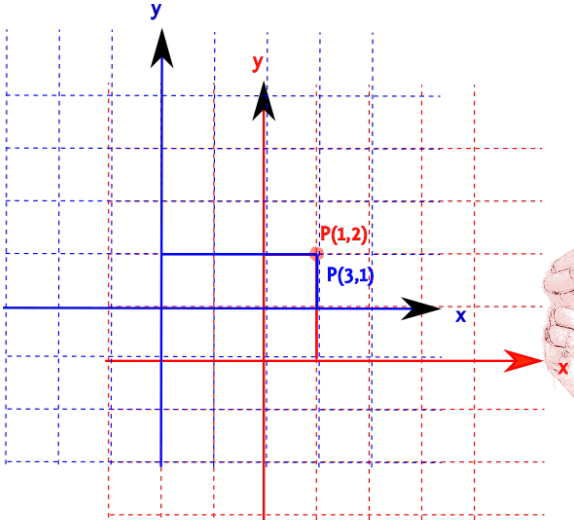
A point P in $\mathbb{R}^2$ space is represented as $P = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

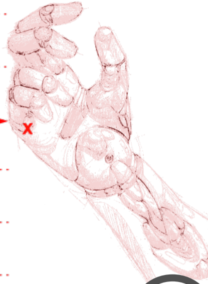


Coordinate systems

The $\mathbb{R}^2$ case

We can have different coordinate systems describing the same points. The coordinate systems might be translated relative to each other

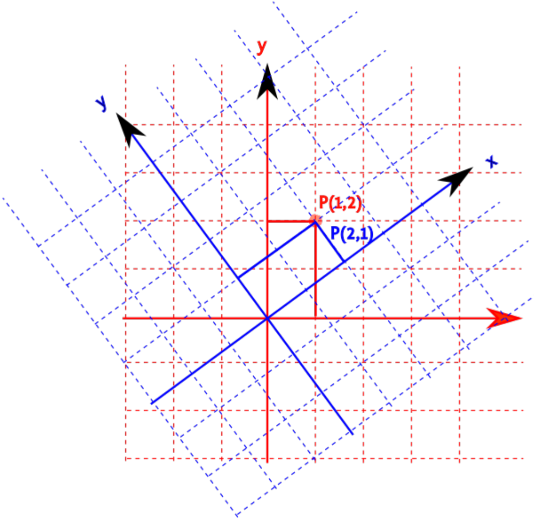




Coordinate systems

The $\mathbb{R}^2$ case

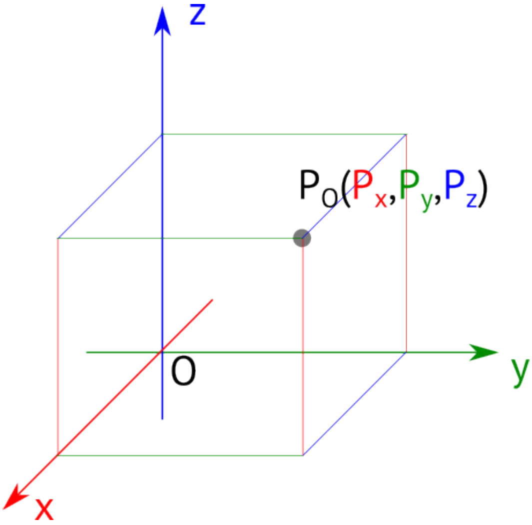
Or they can be rotated relative to each other



Coordinate systems

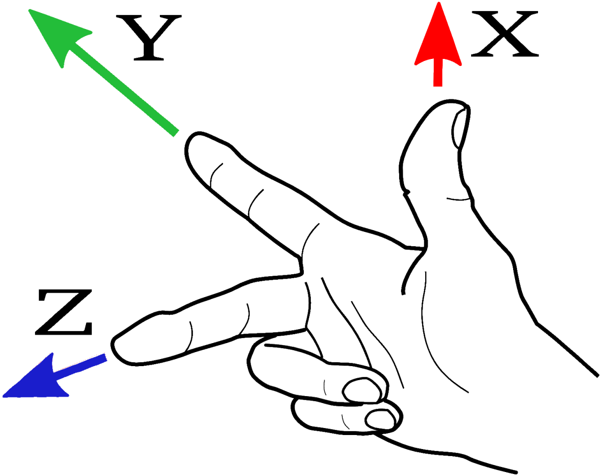
The $\mathbb{R}^3$ case

In three dimensional space (3D), we need three axes to describe the position of each point. Each of these axes must be perpendicular to the other two.



Coordinate systems

The rule of the right hand



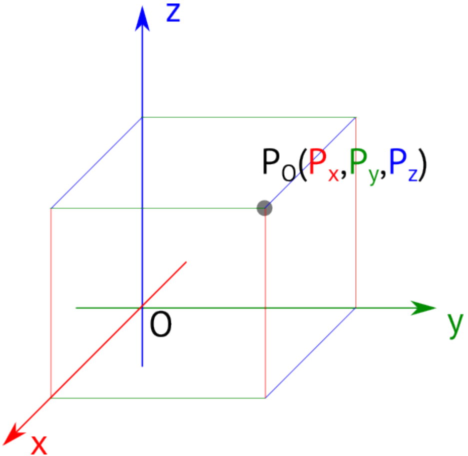
Points

Description of points

Since we might have different coordinate frames defined, we need to define the notation to describe the potision of a point P in respect to a coordinate frame

For a point P described in coordinate frame O, we will use the following notation to describe its position

$$P_O = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$



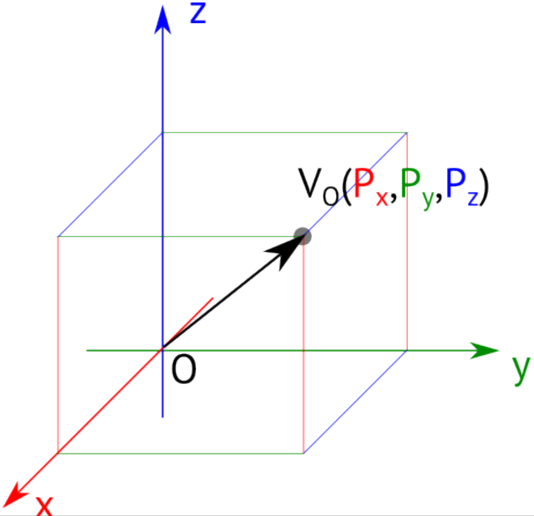
Vectors

Description of vectors

Vectors are just like points!

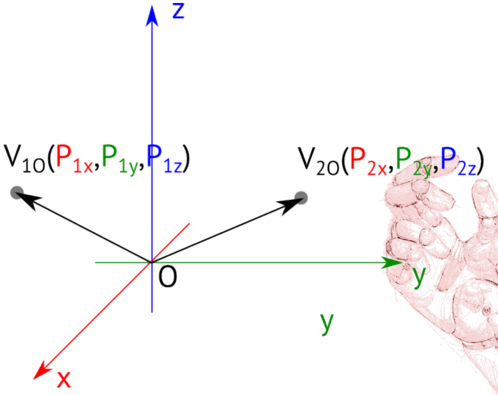
A vector V described in coordinate frame O, is totally defined by its end point P and we use the same notation as points

$$V_O = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$



When we have multiple vectors, we can group them together

$$V_O = \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} P_1x & P_2x \\ P_1y & P_2y \\ P_1z & P_2z \end{bmatrix}$$

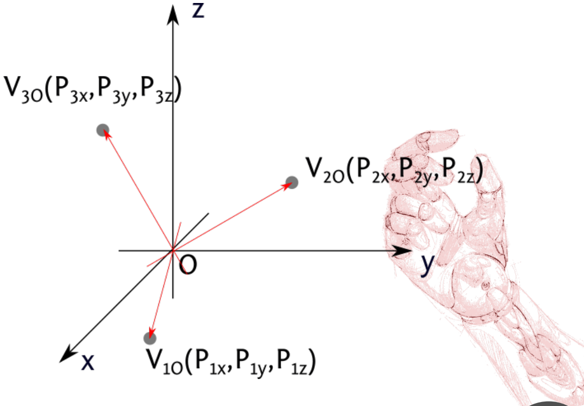


Coordinate frames

Description of coordinate frames

A coordinate system (a.k.a coordinate frame) is a set of three vectors. Therefore, we can describe it in respect to another coordinate frame using the notation we know

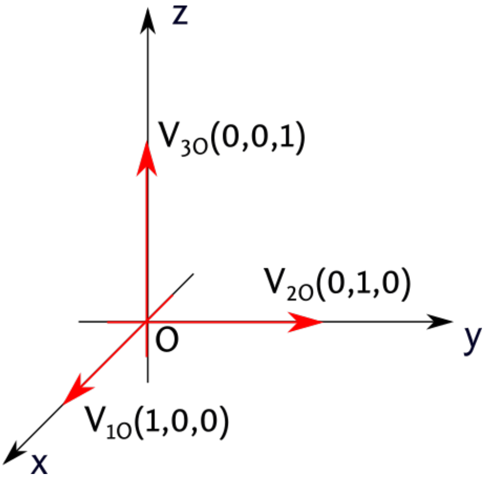
$$V_O = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} P_1x & P_2x & P_3x \\ P_1y & P_2y & P_3y \\ P_1z & P_2z & P_3z \end{bmatrix}$$



Coordinate frames

Description of coordinate frames

In the special case, when the axes of the two coordinate frames are aligned, we end up with....



Coordinate frames

Description of coordinate frames

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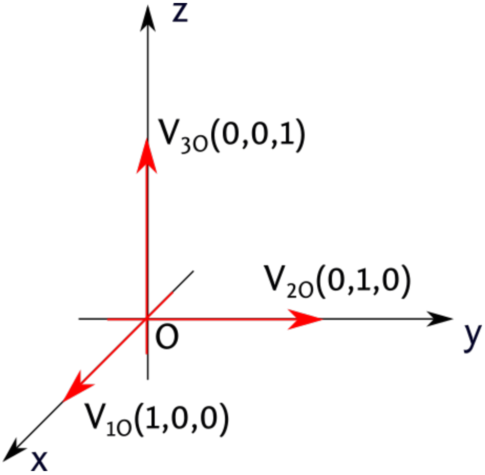


$$V_O = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The identity matrix!

Transformations

Rotations

Robots are about motion, so we need to define a way to move things around. To do this, we use special matrices called *Transformation matrices*.

In 3D, there are three rotations that can be applied, each around one of the three axes.

Rotation around axis:

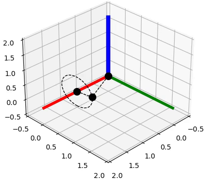
$$R(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

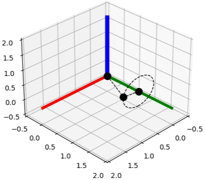
$$R(y,\phi) = \begin{bmatrix} cos\phi & 0 & sin\phi \\ 0 & 1 & 0 \\ -sin\phi & 0 & cos\phi \end{bmatrix}$$

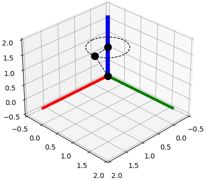
$$R(z,\psi) = \begin{bmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformations

Rotation in 3D





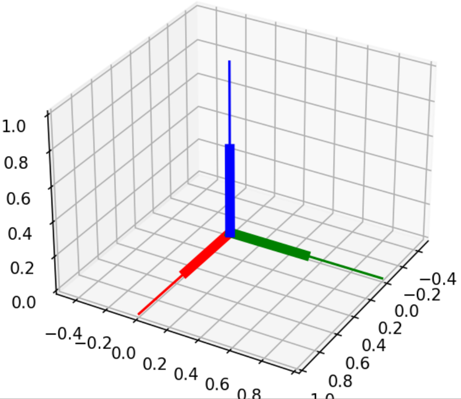


Transformations

Transforming coordinate frames

As we already saw, we use a matrix notation to express a coordinate frame relative to another. A coordinate frame aligned with a basis coordinate frame is expressed with the identity matrix.

$$V_O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

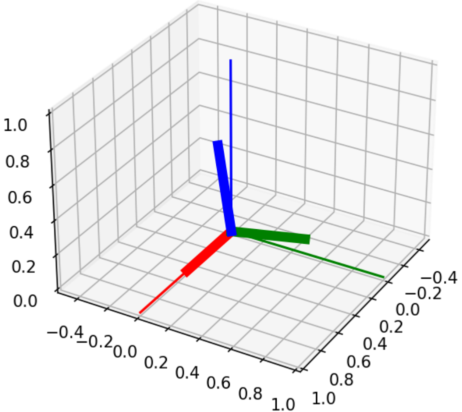


Transformations

Transforming coordinate frames

We can transform the coordinate frame by multiplying its matrix representation with transformation matrices corresponding to the transformation we want.

$$V_O' = Rx(\theta)V_O$$

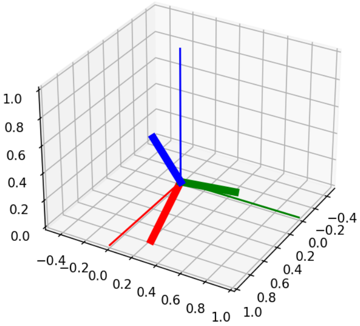


Transformations

Complex transformations

We can apply multiple transformations by multiplying the resulting coordinate frame with a second transformation matrix.

$$V_O' = Ry(\phi)Rx(\theta)V_O$$



Transformations

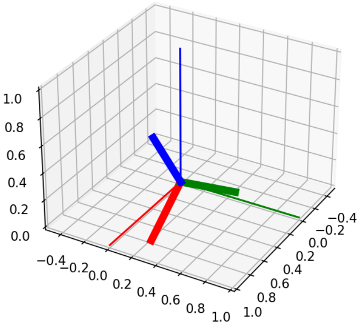
Complex transformations

We can apply multiple transformations by multiplying the resulting coordinate frame with a second transformation matrix.

$$V_O' = Ry(\phi)Rx(\theta)V_O$$

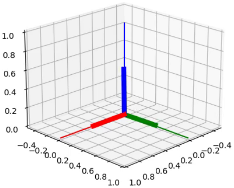
Arbitrary position

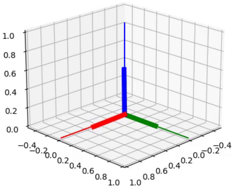
The transition between two arbitrarily positioned coordinate frames can **always** be described in terms of elementary transformations (rotations and translations)

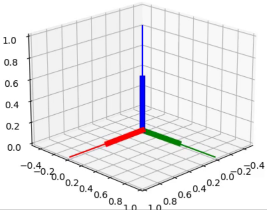


Transformations

Transforming Coordinate Frames







Transformations

What about translation?

The second type of basic transformation is the translation. How do we 'apply' translations to a coordinate frame?

Homogeneous transformation matrix:

$$T = \left[\begin{array}{ccc|c} 3 & \times & 3 & 3\times 1 \\ \hline 1 & \times & 3 & 1\times 1 \end{array}\right] = \left[\begin{array}{ccc|c} & rotation & & \begin{array}{c} trans- \\ la- \\ tion \end{array} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} X_x & Y_x & Z_x & P_x \\ X_y & Y_y & Z_y & P_y \\ X_z & Y_z & Z_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous translations

$$Tx(a) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Ty(b) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Tz(c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformations

Homogeneous translations

$$Tx(a) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

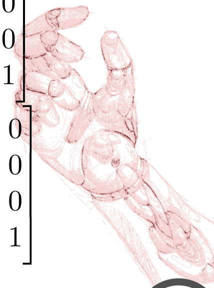
$$Ty(b) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Tz(c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rx(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Ry(\phi) = \begin{bmatrix} cos\phi & 0 & sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -sin\phi & 0 & cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rz(\psi) = \begin{bmatrix} cos\psi & -sin\psi & 0 & 0 \\ sin\omega & cos\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Transformations

Homogeneous transformations

Since the homogeneous matrix for a $\mathbb{R}^3$ transformation is a 4x4 matrix, we need to define vectors as 4×1, in order for the multiplication to be possible. Therefore, coordinate frames are defined as a 4×4 matrix:

$$V_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformations

Multiplication from left

Multiplication of matrices is not commutative!

$$AB \neq BA$$

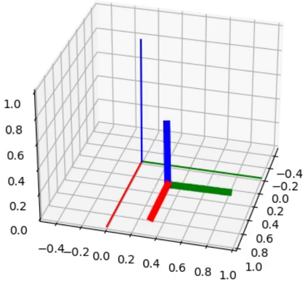
Multiplication from left

Multiplication of matrices is not commutative!

$$AB \neq BA$$

Multiplication from the left results in transformation according to the axes of the base coordinate frame.

$$V_O' = Rx(\theta)V_O$$

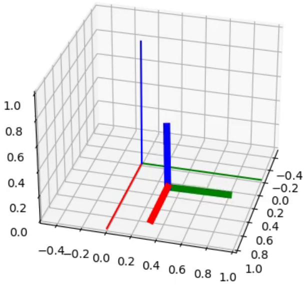


Transformations

Multiplication from right

Multiplication from the right results in transformation according to the axes of the transformed coordinate frame.

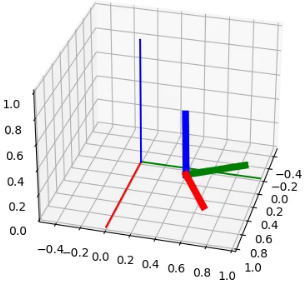
$$V_O' = V_O Rx(\theta)$$



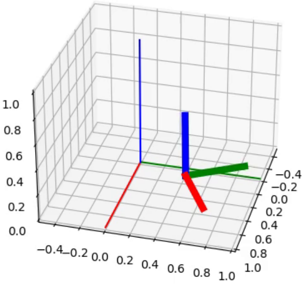
Transformations

Left and right multiplication

$$V_O' = Rx(\theta)V_O$$



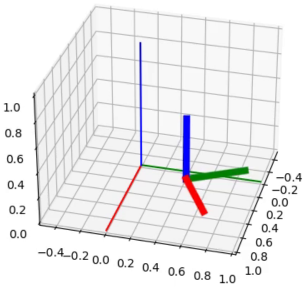
$$V_O' = V_O Rx(\theta)$$



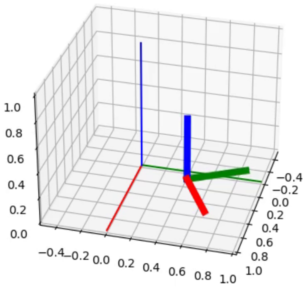
Transformations

Left and right multiplication

$$V_O' = Tx(d)V_O$$



$$V_O' = V_O Tx(d)$$



Arbitrary position

The transition between two arbitrarily positioned coordinate frames can **always** be described in terms of elementary transformations (rotations and translations)

Arbitrary **orientation**

The transition between two arbitrarily **oriented** coordinate frames can **always** be described in terms of **three elementary rotations**

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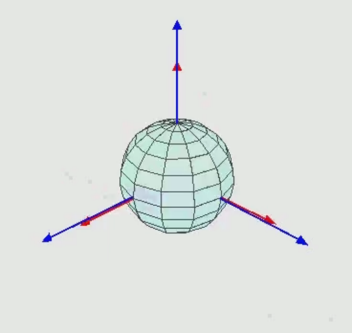
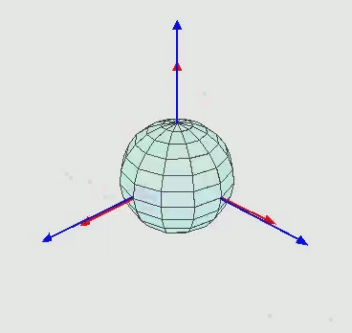


Arbitrary **orientation**

The transition between two arbitrarily **oriented** coordinate frames can **always** be described in terms of **three elementary rotations**



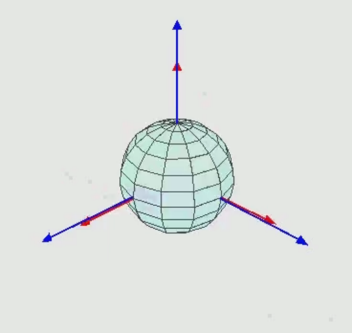
Three subsequent rotations, along specific axes

Transformations

Transforming Coordinate Frames

Arbitrary orientation

The transition between two arbitrarily **oriented** coordinate frames can **always** be described in terms of **three elementary rotations**



Three subsequent rotations, along specific axes

- Z-X-Z
- X-Y-X
- X-Y-Z
- ...

Quaternions

Quaternions is a number system that extends the complex numbers. It can be used to represent relative orientation of two coordinate frames

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

Transformations

Quaternions

Quaternions

Quaternions is a number system that extends the complex numbers. It can be used to represent relative orientation of two coordinate frames

$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$

x	**1**	**i**	**j**	**k**
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Transformations

Quaternions

Quaternions

Quaternions is a number system that extends the complex numbers. It can be used to represent relative orientation of two coordinate frames

$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$

x	**1**	**i**	**j**	**k**
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Trust me, it is weird

Transformations

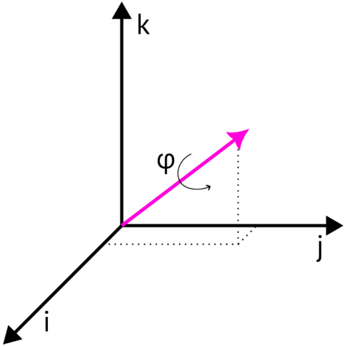
Quaternions

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

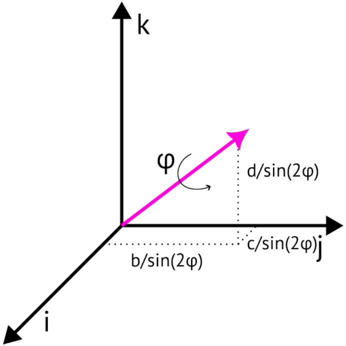
Transformations

Quaternions

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$



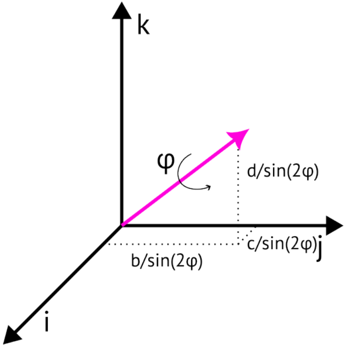
$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$



Transformations

Quaternions

$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$

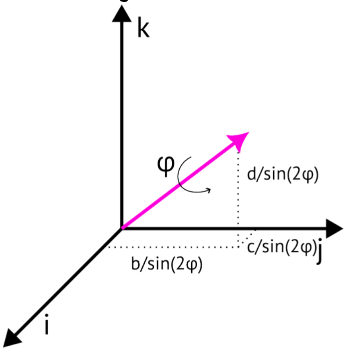


$$p = cos(2\phi) + sin(2\phi)\left(\frac{b}{sin(2\phi)}\mathbf{i} + \frac{c}{sin(2\phi)}\mathbf{j} + \frac{d}{sin(2\phi)}\mathbf{k}\right)$$

Transformations

Quaternions

$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$



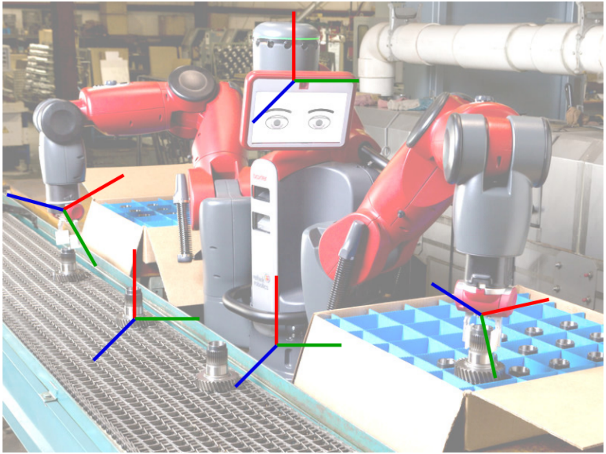
$$p = cos(2\phi) + sin(2\phi)(\frac{b}{sin(2\phi)}\mathbf{i} + \frac{c}{sin(2\phi)}\mathbf{j} + \frac{d}{sin(2\phi)}\mathbf{k})$$

$$p^{'} = qpq^{-}1$$

Switching frames

Because switching is useful

Sometimes, we know the coordinates of a point in one coordinate frame, but we need to describe it in a second frame. This is possible if we know the relative position of the two coordinate frames



Switching frames

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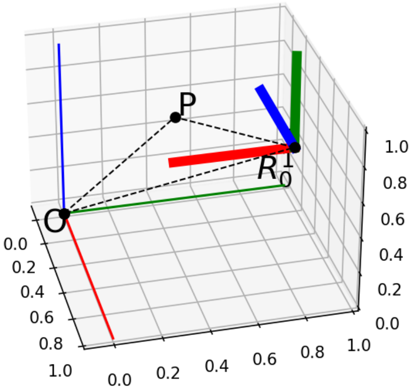
Sometimes, we know the coordinates of a point in one coordinate frame, but we need to describe it in a second frame. This is possible if we know the relative position of the two coordinate frames

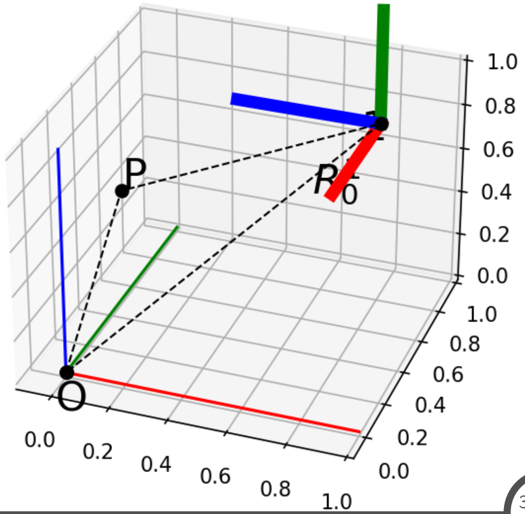
$$P_0 = R_0^1 P_1$$



Switching frames

Example

$$P_0 = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 1 \end{bmatrix}$$

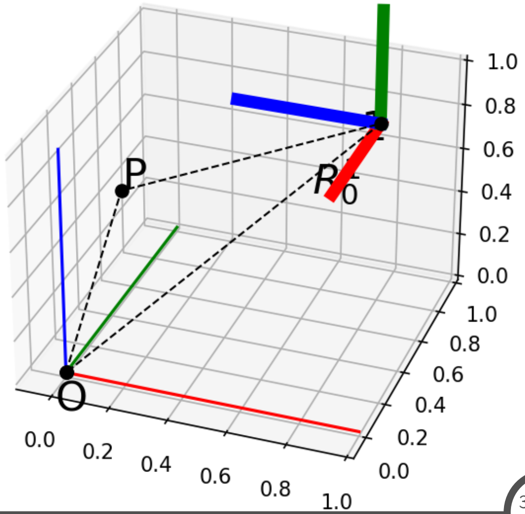


Switching frames

Example

$$P_0 = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 1 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} 0 & -1 & 0 & 0.8 \\ 0 & 0 & 1 & -0.8 \\ -1 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Switching frames

Example

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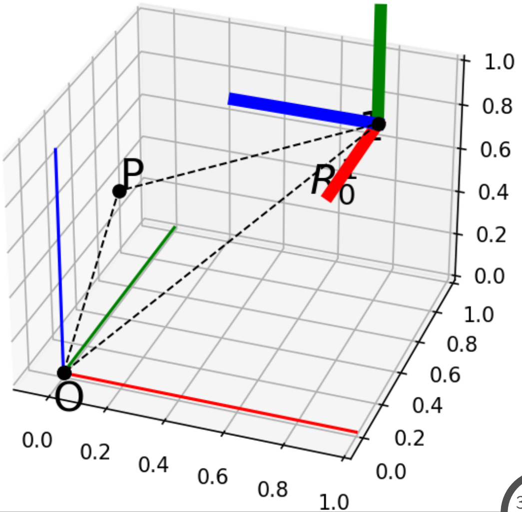
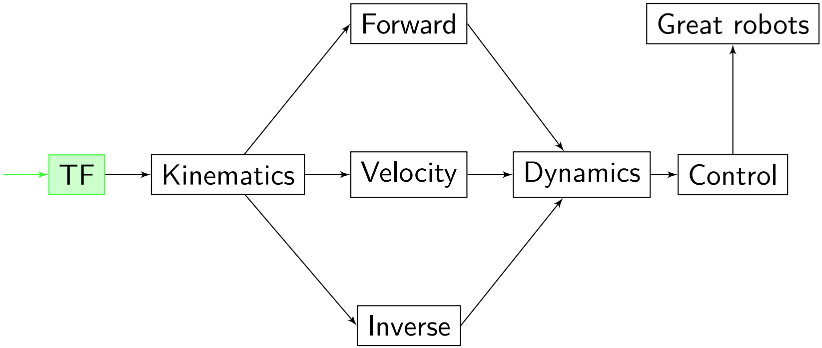
$$R_1^0 = \begin{bmatrix} 0 & -1 & 0 & 0.8 \\ 0 & 0 & 1 & -0.8 \\ -1 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_1 = R_1^0 P_0 = \begin{bmatrix} 0.3 \\ -0.3 \\ 0.8 \\ 1 \end{bmatrix}$$



Grand scheme

The big picture



Questions?