



Inverse kinematics

Cause inverse is better



**TECHNICAL
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Agenda

- Background
- Better understanding of the forward kinematics
- What is inverse kinematics?
- Solving
- Examples



Recap

What we saw last weeks?

DH Modified Parameters

We define each parameter for the length and angles from joint i until the joint $i + 1$

- d_i : Joint offset (length) from joint i to joint $i+1$
- θ_i : Joint angle from joint i to joint $i+1$
- r_i : Link length from joint i to joint $i+1$
- α_i : Link twist (angle) from joint i to joint $i+1$



Recap

What we saw last weeks?

$$R_i^{i+1} = \left[\begin{array}{ccc|c} \cos \theta_i & -\sin \theta_i & 0 & r_i \\ \sin \theta_i \cos \alpha_i & \cos \theta_i \cos \alpha_i & -\sin \alpha_i & -d_i \sin \alpha_i \\ \sin \theta_i \sin \alpha_i & \cos \theta_i \sin \alpha_i & \cos \alpha_i & d_i \cos \alpha_i \\ \hline 0 & 0 & 0 & 1 \end{array} \right]$$

$$R_0^n = R_0^1 * R_1^2 * \dots * R_{n-1}^n$$



Recap

What we saw last weeks?

We can calculate each column of the Jacobian matrix individually. Each column represents one joint. If joint i is revolute, then:

$$J_i = \begin{bmatrix} z_i \times (o_{n+1} - o_i) \\ z_i \end{bmatrix}$$

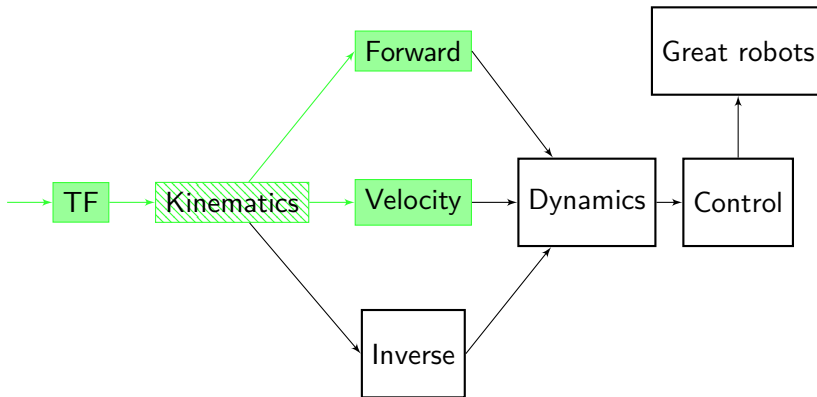
If joint i is prismatic, then:

$$J_i = \begin{bmatrix} z_i \\ 0 \end{bmatrix}$$



Grand scheme

The big picture



Forward kinematics

Revising

Definition

A transformation matrix that calculates the pose of the robot's end effector in terms of the joint coordinates q_1, q_2, \dots, q_n

$$R_0^n(q) = \left[\begin{array}{ccc|c} X_x & Y_x & Z_x & P_x \\ X_y & Y_y & Z_y & P_y \\ X_z & Y_z & Z_z & P_z \\ \hline 0 & 0 & 0 & 1 \end{array} \right]$$



Forward kinematics

Revising

$$R_0^n(q) = \left[\begin{array}{ccc|c} X_x & Y_x & Z_x & P_x \\ X_y & Y_y & Z_y & P_y \\ X_z & Y_z & Z_z & P_z \\ \hline 0 & 0 & 0 & 1 \end{array} \right]$$

How many of these elements are independent?



Forward kinematics

Revising

$$R_0^n(q) = \left[\begin{array}{ccc|c} X_x & Y_x & Z_x & P_x \\ X_y & Y_y & Z_y & P_y \\ X_z & Y_z & Z_z & P_z \\ \hline 0 & 0 & 0 & 1 \end{array} \right]$$

How many of these elements are independent?

$$\sqrt{X_x^2 + X_y^2 + X_z^2} = \sqrt{Y_x^2 + Y_y^2 + Y_z^2} = \sqrt{Z_x^2 + Z_y^2 + Z_z^2} = 1$$



Forward kinematics

Revising

$$R_0^n(q) = \left[\begin{array}{ccc|c} X_x & Y_x & Z_x & P_x \\ X_y & Y_y & Z_y & P_y \\ X_z & Y_z & Z_z & P_z \\ \hline 0 & 0 & 0 & 1 \end{array} \right]$$

How many of these elements are independent?

$$\sqrt{X_x^2 + X_y^2 + X_z^2} = \sqrt{Y_x^2 + Y_y^2 + Y_z^2} = \sqrt{Z_x^2 + Z_y^2 + Z_z^2} = 1$$

$$\begin{bmatrix} X_x \\ X_y \\ X_z \end{bmatrix} \times \begin{bmatrix} Y_x \\ Y_y \\ Y_z \end{bmatrix} = \begin{bmatrix} Z_x \\ Z_y \\ Z_z \end{bmatrix}, \quad \begin{bmatrix} Y_x \\ Y_y \\ Y_z \end{bmatrix} \times \begin{bmatrix} Z_x \\ Z_y \\ Z_z \end{bmatrix} = \begin{bmatrix} X_x \\ X_y \\ X_z \end{bmatrix}$$



Forward kinematics

Better understanding

This is basically a function of $q = [q_1, q_2, \dots, q_n]$ that returns the pose of the end-effector



Forward kinematics

Better understanding

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$$f(q) \mapsto P_x, P_y, P_z, R$$



Forward kinematics

Better understanding

This is basically a function of $q = [q_1, q_2, \dots, q_n]$ that returns the pose of the end-effector

$$f(q) \mapsto P_x, P_y, P_z, R$$

Where R is the orientation defined in terms of X_x, X_y, \dots



Forward kinematics

Better understanding

Since the orientation part has only three degrees of freedom, it can also be expressed as three *Euler* angles:

$$R_z(\psi)R_y(\theta)R_x(\phi) = \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & s_\phi s_\psi + c_\phi c_\psi s_\theta \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$



Forward kinematics

Better understanding

Since the orientation part has only three degrees of freedom, it can also be expressed as three *Euler* angles:

$$R_z(\psi)R_y(\theta)R_x(\phi) = \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & s_\phi s_\psi + c_\phi c_\psi s_\theta \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$

Therefore, the forward kinematics can be expressed as:

$$f(q) \mapsto P_x, P_y, P_z, \phi, \theta, \psi$$



Interlude

UpTown!



Boston Dynamics

Inverse Kinematics

Definition

The inverse kinematics model is the 'inverse' of the forward kinematics model.



Inverse Kinematics

Definition

The inverse kinematics model is the 'inverse' of the forward kinematics model.

$$g(P_x, P_y, P_z, \phi, \theta, \psi) \mapsto q = [q_1, q_2, \dots, q_n]$$



Inverse Kinematics

Definition

The inverse kinematics model is the 'inverse' of the forward kinematics model.

$$g(P_x, P_y, P_z, \phi, \theta, \psi) \mapsto q = [q_1, q_2, \dots, q_n]$$

A function calculates the joint coordinates for a specific pose.



Inverse and Forward kinematics

What is the difference?

Forward kinematics

I want to know where will my end-effector be, if I give specific coordinates (values) to each joint



Inverse and Forward kinematics

What is the difference?

Forward kinematics

I want to know where will my end-effector be, if I give specific coordinates (values) to each joint

Inverse kinematics

I want to know what should the joint coordinates (values) be in order for my end-effector to reach a specific pose



Inverse and Forward kinematics

What is the difference?

For robotics applications, the inverse model is way more useful.



Inverse and Forward kinematics

What is the difference?

For robotics applications, the inverse model is way more useful.

Can you understand why?



Inverse and Forward kinematics

What is the difference?

For robotics applications, the inverse model is way more useful.

But it is also most difficult to derive and we need the forward kinematics to derive it.



Inverse and Forward kinematics

What is the difference?

For robotics applications, the inverse model is way more useful.

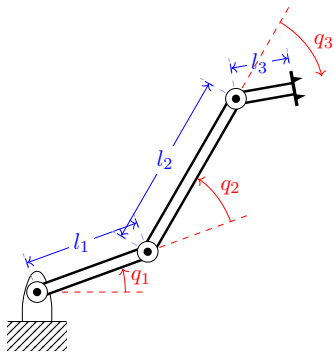
But it is also most difficult to derive and we need the forward kinematics to derive it.

Can you understand why?

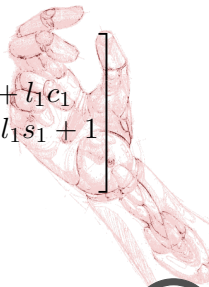


Inverse kinematics model

Why so difficult?

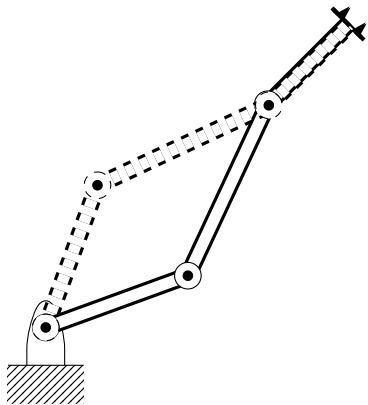


$$R_0^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{1,2,3} & -s_{1,2,3} & l_2 c_{1,2} + l_3 c_{1,2,3} + l_1 c_1 \\ 0 & s_{1,2,3} & c_{1,2,3} & l_2 s_{1,2} + l_3 s_{1,2,3} + l_1 s_1 + 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Inverse kinematics model

Why so difficult?



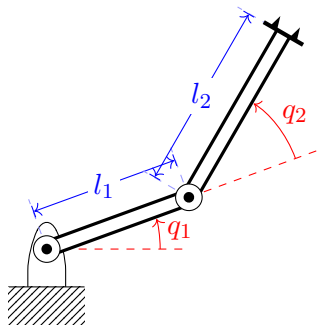
The inverse kinematic model might have more than one solution for a specific robot pose



Inverse kinematics model

Derivation

The inverse model can be difficult to solve even for simple models



$$R(q_1, q_2) = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Inverse kinematics model

Derivation

$$\begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X_X & Y_X & Z_X & P_x \\ X_Y & Y_Y & Z_Y & P_y \\ X_Z & Y_Z & Z_Z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Inverse kinematics model

Derivation

$$\begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X_X & Y_X & Z_X & P_x \\ X_Y & Y_Y & Z_Y & P_y \\ X_Z & Y_Z & Z_Z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & s_\phi s_\psi + c_\phi c_\psi s_\theta & P_x \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi & P_y \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Inverse kinematics model

Derivation

$$\begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X_X & Y_X & Z_X & P_x \\ X_Y & Y_Y & Z_Y & P_y \\ X_Z & Y_Z & Z_Z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos(q_1 + q_2) = X_x = Y_y = c_\theta c_\psi = \\ c_\phi c_\psi + s_\theta s_\psi s_\psi$$

$$\sin(q_1 + q_2) = X_y = -Y_x = c_\theta s_\psi = \\ -c_\psi s_\theta s_\phi - c_\phi s_\psi$$

$$l_2 \cos(q_1 + q_2) + l_1 \cos q_1 = P_x$$

$$l_2 \sin(q_1 + q_2) + l_1 \sin q_1 = P_y$$

$$0 = P_z$$



Inverse kinematics model

Derivation

$$\begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X_X & Y_X & Z_X & P_x \\ X_Y & Y_Y & Z_Y & P_y \\ X_Z & Y_Z & Z_Z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$l_2 \cos(q_1 + q_2) + l_1 \cos q_1 = P_x$$

$$l_2 \sin(q_1 + q_2) + l_1 \sin q_1 = P_y$$

$$0 = P_z$$



Inverse kinematics model

Derivation

$$\begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X_X & Y_X & Z_X & P_x \\ X_Y & Y_Y & Z_Y & P_y \\ X_Z & Y_Z & Z_Z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos(q_1 + q_2) = X_x = Y_y = c_\theta c_\psi = \\ c_\phi c_\psi + s_\theta s_\psi s_\psi$$

$$\sin(q_1 + q_2) = X_y = -Y_x = c_\theta s_\psi = \\ -c_\psi s_\theta s_\phi - c_\phi s_\psi$$

$$l_2 \cos(q_1 + q_2) + l_1 \cos q_1 = P_x$$

$$l_2 \sin(q_1 + q_2) + l_1 \sin q_1 = P_y$$

$$0 = P_z$$

How much 'freedom' do we have?



Inverse kinematics model

Derivation

$$\begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X_X & Y_X & Z_X & P_x \\ X_Y & Y_Y & Z_Y & P_y \\ X_Z & Y_Z & Z_Z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos(q_1 + q_2) = X_x = Y_y = c_\theta c_\psi = \\ c_\phi c_\psi + s_\theta s_\psi s_\psi$$

$$\sin(q_1 + q_2) = X_y = -Y_x = c_\theta s_\psi = \\ -c_\psi s_\theta s_\phi - c_\phi s_\psi$$

$$l_2 \cos(q_1 + q_2) + l_1 \cos q_1 = P_x$$

$$l_2 \sin(q_1 + q_2) + l_1 \sin q_1 = P_y$$

$$0 = P_z$$

How much 'freedom' do we have?
How do we solve this?



Inverse kinematics model

Derivation

We are looking for this

$$q_1 = f(P_x, P_y)$$

$$q_2 = g(P_x, P_y)$$



Inverse kinematics model

Derivation

We are looking for this

$$q_1 = f(P_x, P_y)$$

$$q_2 = g(P_x, P_y)$$

- Analytical solutions



Inverse kinematics model

Derivation

We are looking for this

$$q_1 = f(P_x, P_y)$$

$$q_2 = g(P_x, P_y)$$

- Analytical solutions
- Geometric solutions



Inverse kinematics model

Derivation

We are looking for this

$$q_1 = f(P_x, P_y)$$

$$q_2 = g(P_x, P_y)$$

- Analytical solutions
- Geometric solutions
- Numerical solutions



Inverse kinematics model

Analytical solutions

The idea is to try to find an equation for each joint variable q_n that only depends on the pose or on other joint variables that have already been expressed in terms of pose.



Inverse kinematics model

Analytical solutions

The idea is to try to find an equation for each joint variable q_n that only depends on the pose or on other joint variables that have already been expressed in terms of pose.

- We equate the forward kinematics with the general homogeneous matrix



Inverse kinematics model

Analytical solutions

The idea is to try to find an equation for each joint variable q_n that only depends on the pose or on other joint variables that have already been expressed in terms of pose.

- We equate the forward kinematics with the general homogeneous matrix
- We identify joint variables that can be isolated



Inverse kinematics model

Analytical solutions

The idea is to try to find an equation for each joint variable q_n that only depends on the pose or on other joint variables that have already been expressed in terms of pose.

- We equate the forward kinematics with the general homogeneous matrix
- We identify joint variables that can be isolated
- We identify pair of joint variables that can be simplified by division



Inverse kinematics model

Analytical solutions

The idea is to try to find an equation for each joint variable q_n that only depends on the pose or on other joint variables that have already been expressed in terms of pose.

- We equate the forward kinematics with the general homogeneous matrix
- We identify joint variables that can be isolated
- We identify pair of joint variables that can be simplified by division
- We identify pair of joint variables that can be simplified by trigonometry



Inverse kinematics model

Analytical solutions

If not all joint variables are expressed as a function of the pose, we multiply from left(right) the inverse transformation of the first(last) joint.



Inverse kinematics model

Analytical solutions

If not all joint variables are expressed as a function of the pose, we multiply from left(right) the inverse transformation of the first(last) joint.

$$R_0^n = R_0^1 R_1^2 \dots R_{n-1}^n = R_g$$



Inverse kinematics model

Analytical solutions

If not all joint variables are expressed as a function of the pose, we multiply from left(right) the inverse transformation of the first(last) joint.

$$R_0^n = R_0^1 R_1^2 \dots R_{n-1}^n = R_g$$

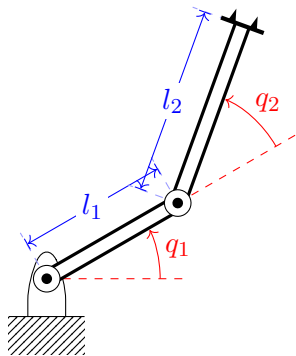
$$(R_0^1)^{-1} R_0^n = (R_0^1)^{-1} R_g \text{ or } R_0^n (R_{n-1}^n)^{-1} = R_g (R_{n-1}^n)^{-1}$$

And we try to isolate again



Inverse kinematics model

Examples



$$R(q_1, q_2) = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$

$$y = l_2 s_{1,2} + l_1 s_1$$



Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$

$$y = l_2 s_{1,2} + l_1 s_1$$

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 (c_1 c_{1,2} + s_1 s_{1,2})$$



Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$

$$y = l_2 s_{1,2} + l_1 s_1$$

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 (c_1 c_{1,2} + s_1 s_{1,2})$$

Remember:

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$



Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$

$$y = l_2 s_{1,2} + l_1 s_1$$

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 (c_1 c_{1,2} + s_1 s_{1,2})$$

Remember:

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

Therefore:

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 c_2$$



Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$

$$y = l_2 s_{1,2} + l_1 s_1$$

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 (c_1 c_{1,2} + s_1 s_{1,2})$$

Remember:

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

Therefore:

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 c_2$$

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

$$q_2 = \cos^{-1} \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$



Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$

$$y = l_2 s_{1,2} + l_1 s_1$$



Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$

$$y = l_2 s_{1,2} + l_1 s_1$$

Remember:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \cos\alpha\sin\beta + \cos\beta\sin\alpha$$



Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$

$$y = l_2 s_{1,2} + l_1 s_1$$

Remember:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \cos\alpha\sin\beta + \cos\beta\sin\alpha$$

Therefore:

$$x = l_2 c_1 c_2 - l_2 s_1 s_2 + l_1 c_1$$

$$y = l_2 c_1 s_2 + l_2 c_2 s_1 + l_1 s_1$$



Inverse kinematics model

Analytical solution

We can write:

$$x = k_1 c_1 - k_2 s_1$$

$$y = k_1 s_1 + k_2 c_1$$



Inverse kinematics model

Analytical solution

We can write:

$$x = k_1 c_1 - k_2 s_1$$

$$y = k_1 s_1 + k_2 c_1$$

Where:

$$k_1 = l_1 + l_2 c_2$$

$$k_2 = l_2 s_2$$



Inverse kinematics model

Analytical solution

We can write:

$$x = k_1 c_1 - k_2 s_1$$

$$y = k_1 s_1 + k_2 c_1$$

Where:

$$k_1 = l_1 + l_2 c_2$$

$$k_2 = l_2 s_2$$

We define:

$$r = \sqrt{k_1^2 + k_2^2}$$

$$\beta = \text{atan2}(k_2, k_1)$$



Inverse kinematics model

Analytical solution

We can write:

$$x = k_1 c_1 - k_2 s_1$$

$$y = k_1 s_1 + k_2 c_1$$

Where:

$$k_1 = l_1 + l_2 c_2$$

$$k_2 = l_2 s_2$$

We define:

$$r = \sqrt{k_1^2 + k_2^2}$$

$$\beta = \text{atan2}(k_2, k_1)$$

Then:

$$k_1 = r \cos \beta$$

$$k_2 = r \sin \beta$$



Inverse kinematics model

Analytical solution

We can write:

$$x = k_1 c_1 - k_2 s_1$$

$$y = k_1 s_1 + k_2 c_1$$

Where:

$$k_1 = l_1 + l_2 c_2$$

$$k_2 = l_2 s_2$$

We define:

$$r = \sqrt{k_1^2 + k_2^2}$$

$$\beta = \text{atan2}(k_2, k_1)$$

Then:

$$k_1 = r \cos \beta$$

$$k_2 = r \sin \beta$$

So we can write:

$$\frac{x}{r} = \cos \beta c_1 - \sin \beta s_1$$

$$\frac{y}{r} = \cos \beta s_1 + \sin \beta c_1$$



Inverse kinematics model

Analytical solution

We can write:

$$x = k_1 c_1 - k_2 s_1$$

$$y = k_1 s_1 + k_2 c_1$$

Where:

$$k_1 = l_1 + l_2 c_2$$

$$k_2 = l_2 s_2$$

We define:

$$r = \sqrt{k_1^2 + k_2^2}$$

$$\beta = \text{atan2}(k_2, k_1)$$

Then:

$$k_1 = r \cos \beta$$

$$k_2 = r \sin \beta$$

So we can write:

$$\frac{x}{r} = \cos \beta c_1 - \sin \beta s_1$$

$$\frac{y}{r} = \cos \beta s_1 + \sin \beta c_1$$

Or:

$$\frac{x}{r} = \cos(\beta + q_1)$$

$$\frac{y}{r} = \sin(\beta + q_1)$$



Inverse kinematics model

Analytical solution

Are we there yet??



Inverse kinematics model

Analytical solution

Are we there yet??

$$\frac{x}{r} = \cos(\beta + q_1)$$
$$\frac{y}{r} = \sin(\beta + q_1)$$



Inverse kinematics model

Analytical solution

Are we there yet??

$$\frac{x}{r} = \cos(\beta + q_1)$$
$$\frac{y}{r} = \sin(\beta + q_1)$$

$$\beta + q_1 = \operatorname{atan2}\left(\frac{y}{r}, \frac{x}{r}\right) = \operatorname{atan2}(y, x)$$



Inverse kinematics model

Analytical solution

Are we there yet??

$$\frac{x}{r} = \cos(\beta + q_1)$$
$$\frac{y}{r} = \sin(\beta + q_1)$$

$$\beta + q_1 = \operatorname{atan2}\left(\frac{y}{r}, \frac{x}{r}\right) = \operatorname{atan2}(y, x)$$

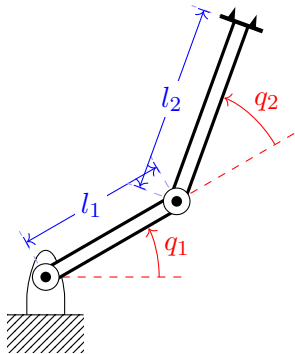
$$q_1 = \operatorname{atan2}(x, y) - \beta = \operatorname{atan2}(y, x) - \operatorname{atan2}(k_2, k_1)$$



Inverse kinematics

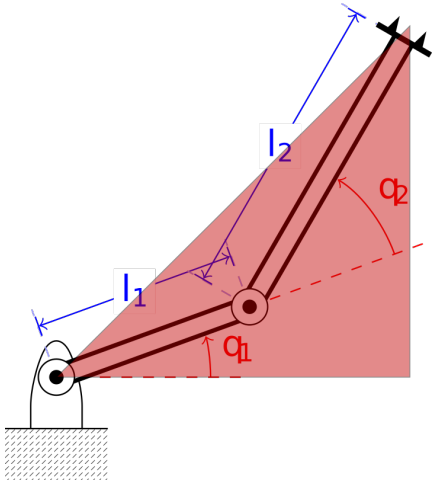
Geometric solutions

We could have solved the previous problem using geometry



Inverse kinematics

Geometric solutions

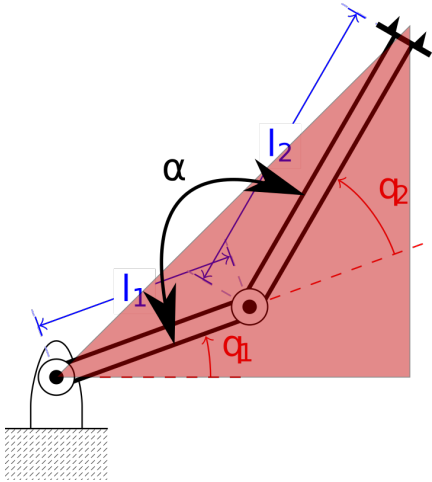


We define the length r using Pythagoras:
$$r^2 = x^2 + y^2$$



Inverse kinematics

Geometric solutions

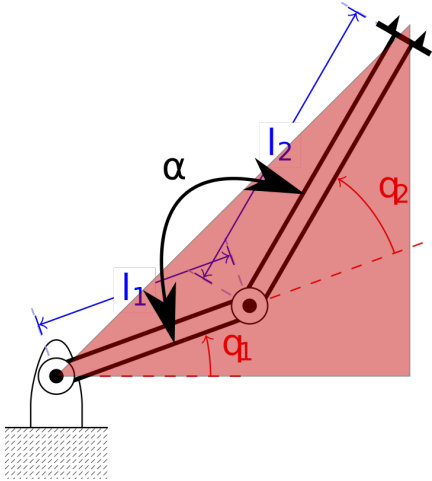


We calculate the angle α using the cosine law:



Inverse kinematics

Geometric solutions



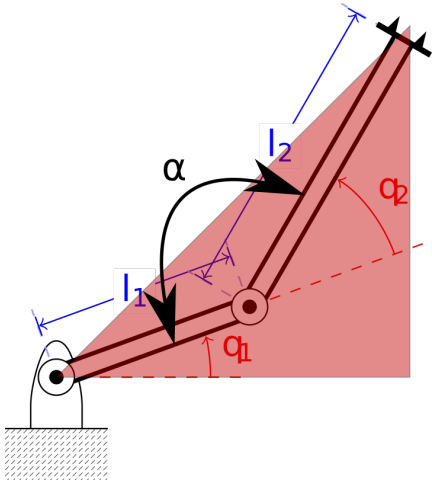
We calculate the angle α using the cosine law:

$$r^2 = l_1^2 + l_2^2 - 2l_1l_2\cos\alpha$$



Inverse kinematics

Geometric solutions



We calculate the angle α using the cosine law:

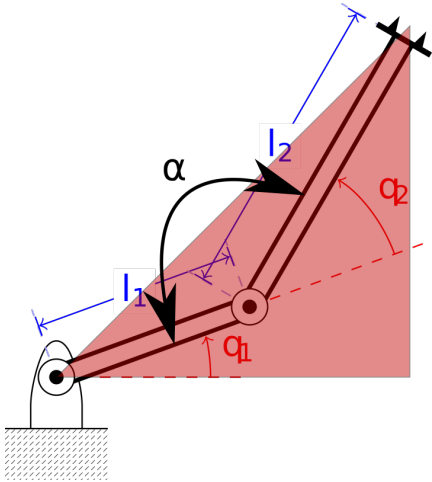
$$r^2 = l_1^2 + l_2^2 - 2l_1l_2\cos\alpha$$

$$\cos\alpha = \frac{l_1^2 + l_2^2 - r^2}{2l_1l_2}$$



Inverse kinematics

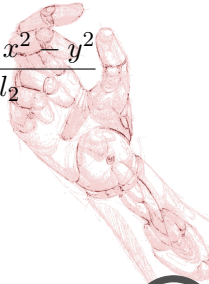
Geometric solutions



We calculate the angle α using the cosine law:

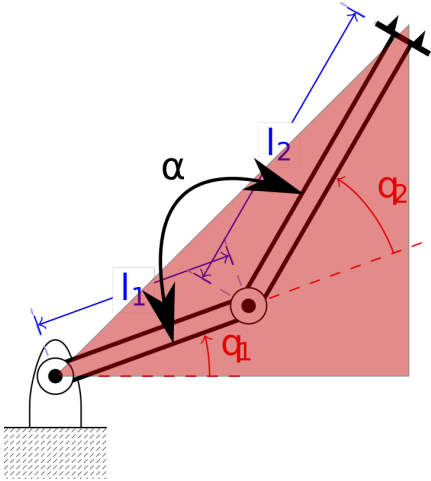
$$r^2 = l_1^2 + l_2^2 - 2l_1l_2\cos\alpha$$

$$\cos\alpha = \frac{l_1^2 + l_2^2 - r^2}{2l_1l_2} \quad \cos\alpha = \frac{l_1^2 + l_2^2 - x^2 - y^2}{2l_1l_2}$$



Inverse kinematics

Geometric solutions

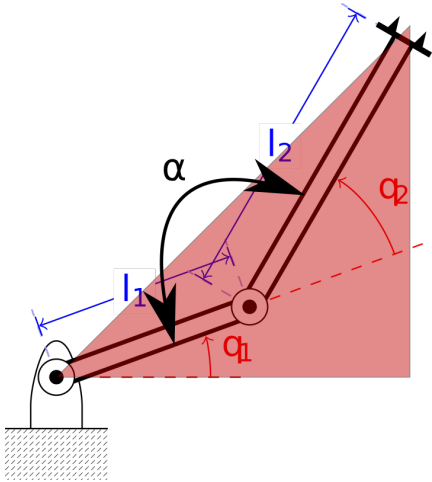


We know that: $\alpha = \pi - q_2$



Inverse kinematics

Geometric solutions



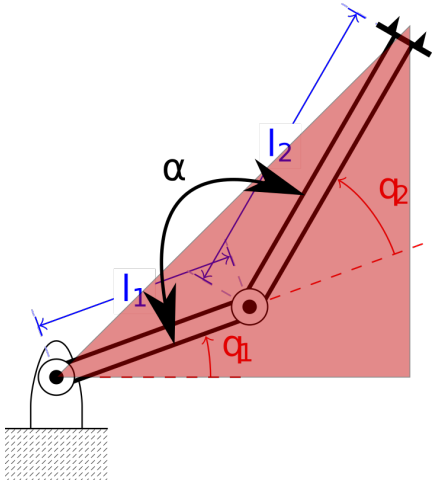
We know that: $\alpha = \pi - q_2$

And: $\cos(\pi - q) = -\cos q$



Inverse kinematics

Geometric solutions



We know that: $\alpha = \pi - q_2$

And: $\cos(\pi - q) = -\cos q$

Therefore:

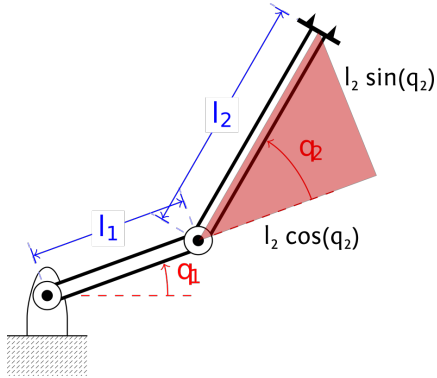
$$\cos q_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

$$q_2 = \cos^{-1} \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$



Inverse kinematics

Geometric solutions

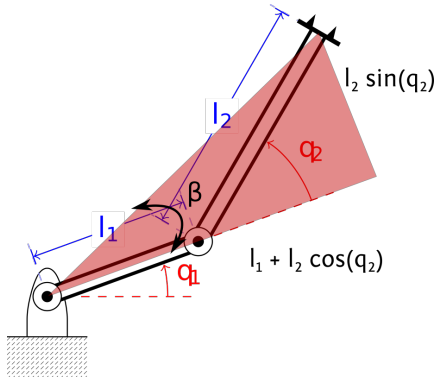


We know that the base of this triangle is $l_2 \cos q_2$ while the height of the triangle is $l_2 \sin q_2$



Inverse kinematics

Geometric solutions



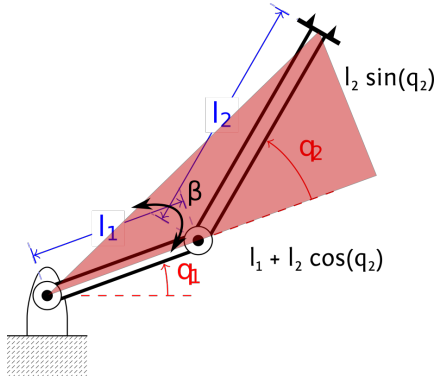
For this triangle, we can calculate the angle β :

$$\beta = \tan^{-1} \frac{l_2 \sin q_2}{l_1 + l_2 \cos q_2}$$



Inverse kinematics

Geometric solutions



For this triangle, we can calculate the angle β :

$$\beta = \tan^{-1} \frac{l_2 \sin q_2}{l_1 + l_2 \cos q_2}$$

From our initial triangle we know that:

$$\tan^{-1} \frac{y}{x} = \beta + q_1$$

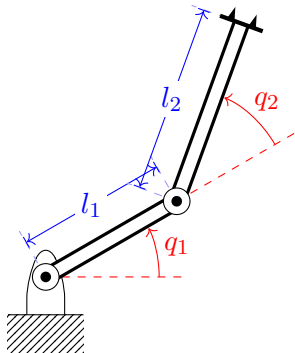


Inverse kinematics

Geometric solutions

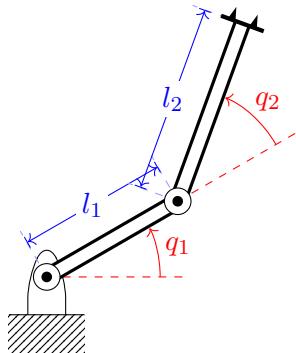
We therefore have:

$$q_1 = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{l_2 \sin q_2}{l_1 + l_2 \cos q_2}$$



Inverse kinematics

Geometric solutions

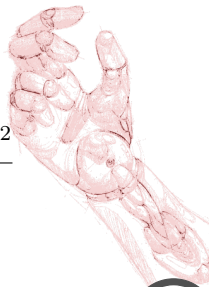


We therefore have:

$$q_1 = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{l_2 \sin q_2}{l_1 + l_2 \cos q_2}$$

And we already know the relationship of q_2 in terms of the position:

$$q_2 = \cos^{-1} \frac{l_1^2 + l_2^2 - x^2 - y^2}{2l_1 l_2}$$



Inverse kinematics model

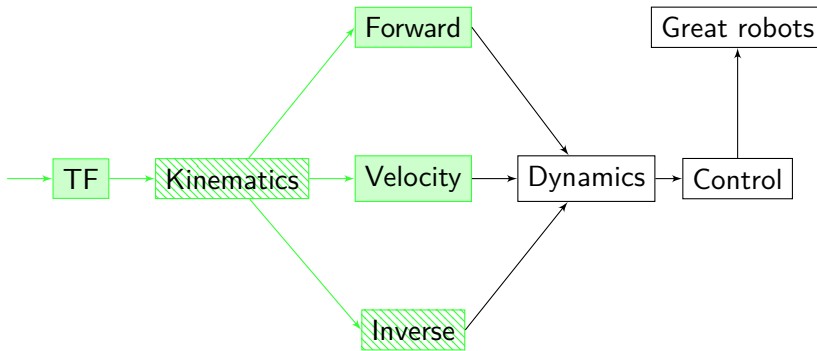
Numerical solutions

Optimisation methods, sem 2, 4th year!



Grand scheme

The big picture





Questions?