Robot Velocity

We have the need for speed

TECHNICAL UNIVERSITY OF CLUJ-NAPOCA ROMANIA

Last update: October 21, 2024

Tassos Natsakis tassos.natsakis@aut.utcluj.ro

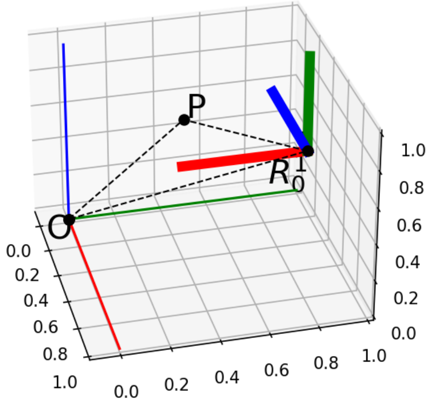
Agenda

- Background
- Linear and angular velocity
- The Jacobian
- Inverting the Jacobian - Singularities
- Velocity ellipsoid

Recap

What do we know already?

$$P_0 = R_0^1 * P_1$$



Recap

What we know already?

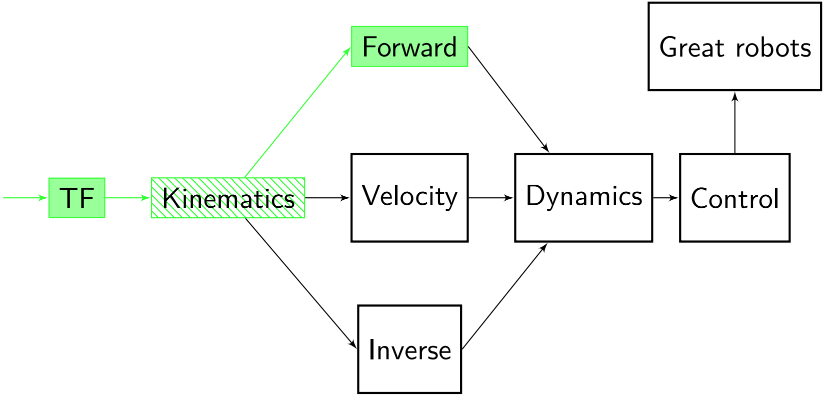
Definition

A transformation matrix that calculates the pose of the robot's end effector in terms of the joint coordinates $q_1, q_2, \ldots, q_n$

$$\left[\begin{array}{ccc|c} X_X & Y_X & Z_X & P_x \\ X_Y & Y_Y & Z_Y & P_y \\ X_Z & Y_Z & Z_Z & P_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Grand scheme

The big picture



Robot velocity

Background

A robot is a mechanism which consists of joints and links.

Robot velocity

Background

A robot is a mechanism which consists of joints and links.

By controlling the position of the joints, we can control the position of the end-effector.

Robot velocity
Background

A robot is a mechanism which consists of joints and links.

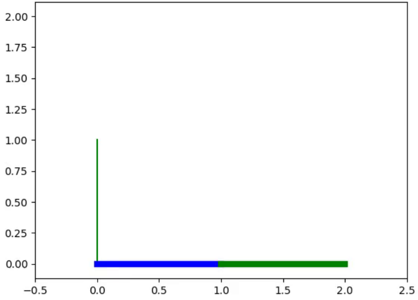
By controlling the position of the joints, we can control the position of the end-effector.

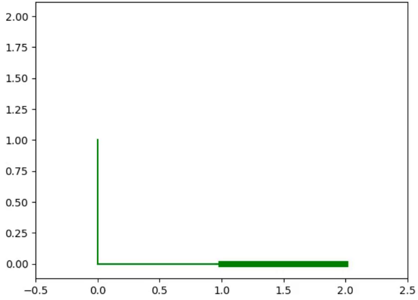
Can we do this for velocities as well?

Linear and angular velocity

What is the difference?

Each of the robot segments can be moving with a linear, angular velocity, or a complex velocity.

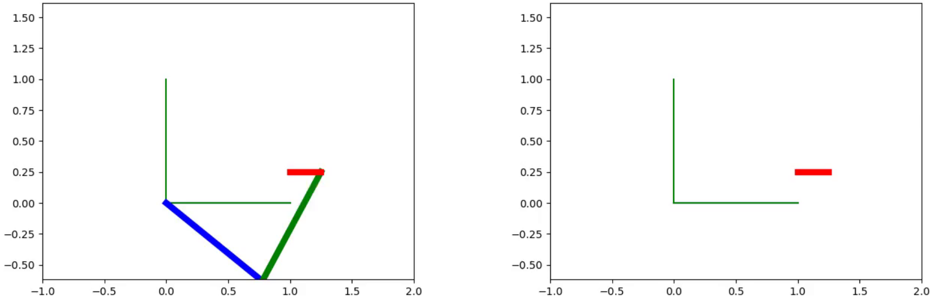




Linear and angular velocity

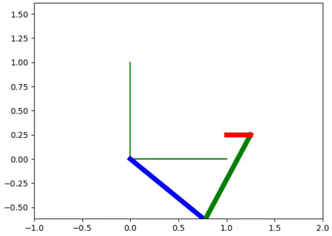
What is the difference?

Each of the robot segments can be moving with a linear, angular or a complex velocity.

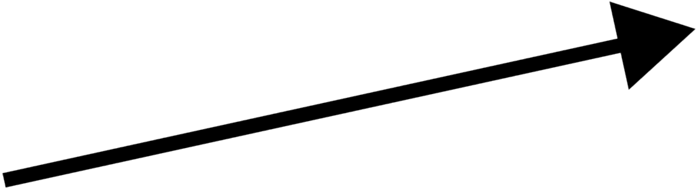


Linear and angular velocity

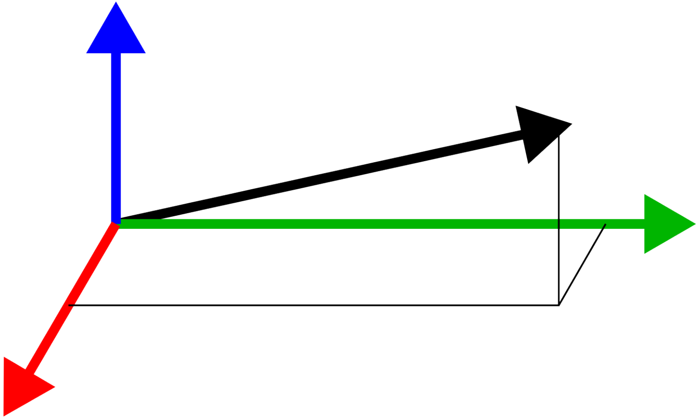
What is the difference?



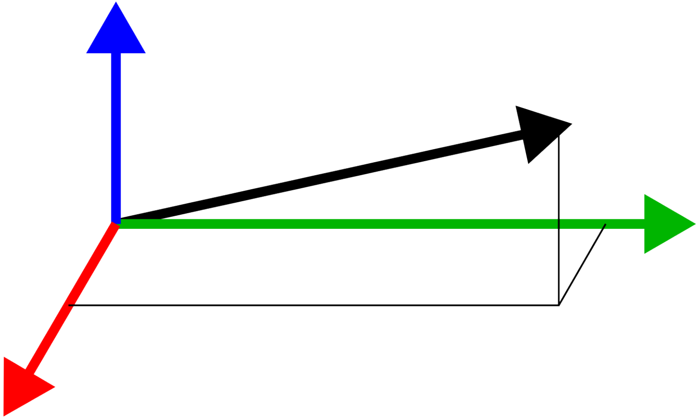
What is linear velocity



What is linear velocity

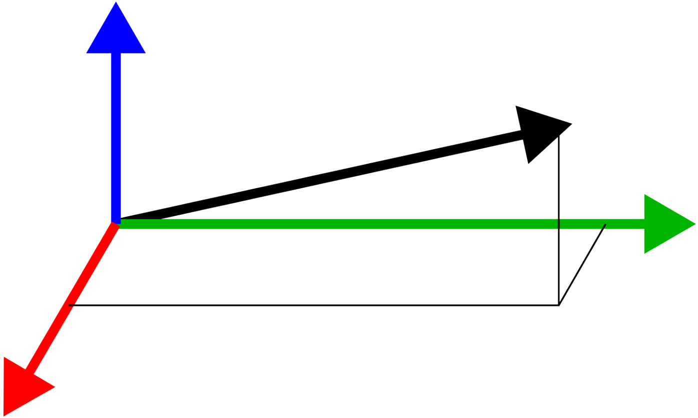


What is linear velocity



$$\vec{u} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

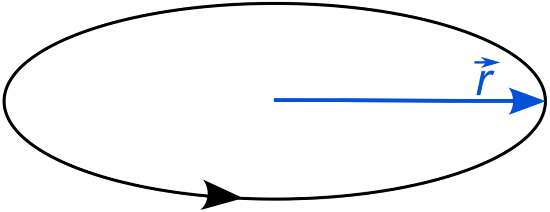
What is linear velocity



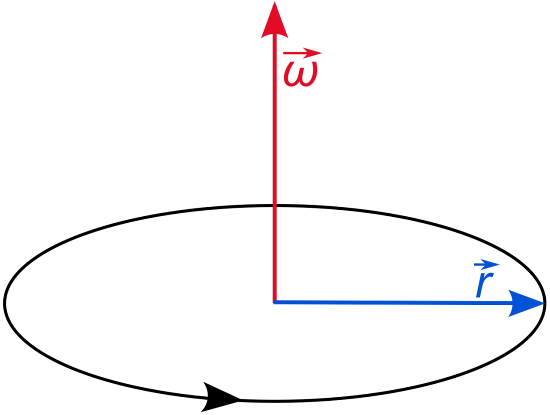
$$\vec{u} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

How do we add up linear velocities?

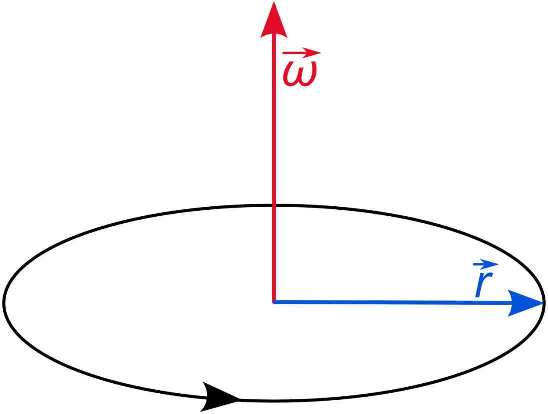
What is angular velocity



What is angular velocity

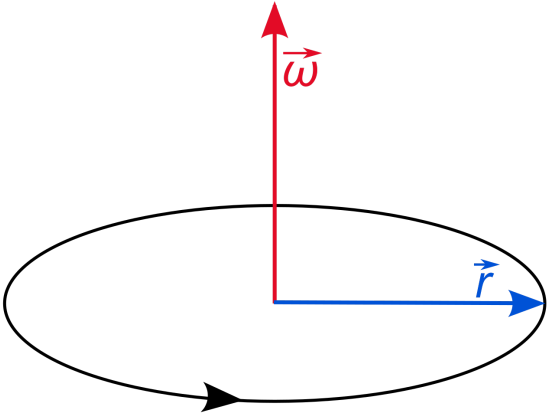


What is angular velocity



$$\vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

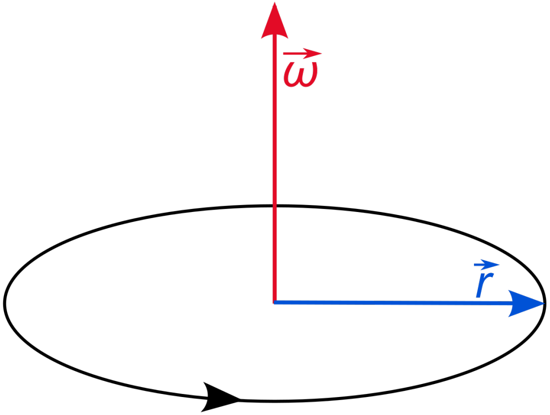
What is angular velocity



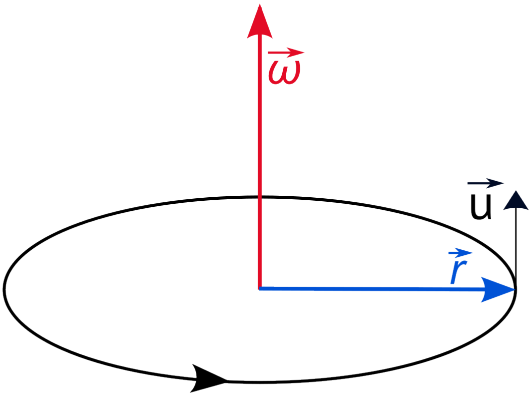
$$\vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

How do we add up angular velocities?

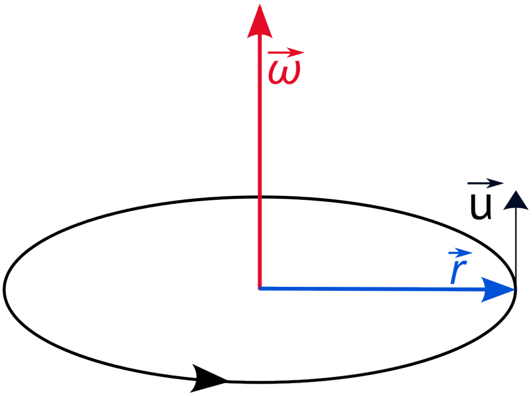
Linear velocity due to rotation



Linear velocity due to rotation



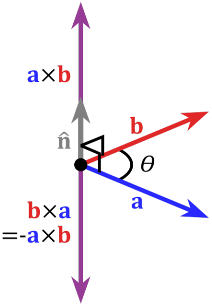
Linear velocity due to rotation



$$\vec{u} = \vec{\omega} \times \vec{r}$$

Defining the Jacobian

What is the cross product?



If we have two vectors a and b with coordinates $[a_i, a_2, a_3]$ and $[b_1, b_2, b_3]$ respectively then, the cross product is defined as:

$$a \times b = (a_2 b_3 - a_3 b_2)i + (a_3 b_1 - a_1 b_3)j + (a_1 b_2 - a_2 b_1)k$$

Robot velocity

End-effector velocity

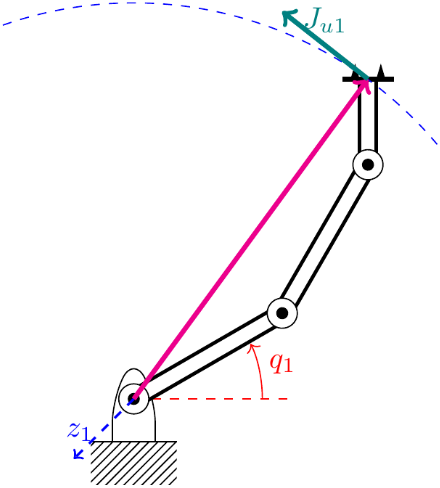
$$u = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\xi = \begin{bmatrix} u \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

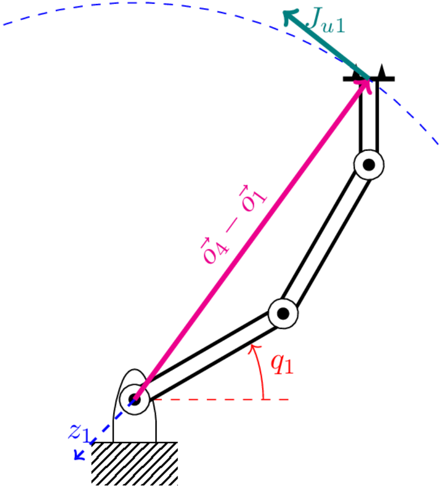
Joint velocity contribution

Linear velocity due to rotation



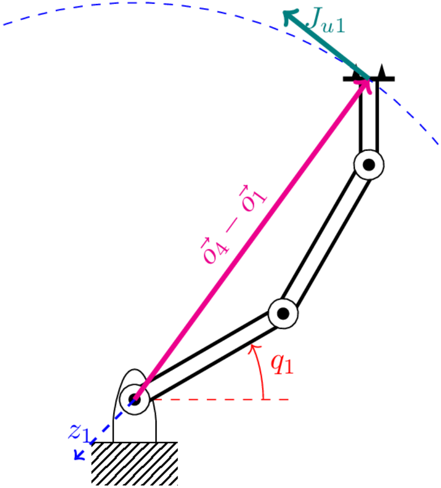
Joint velocity contribution

Linear velocity due to rotation



Joint velocity contribution

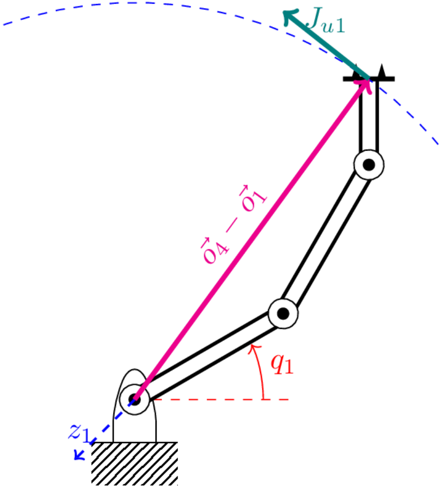
Linear velocity due to rotation



$$\vec{u_1} = \vec{\dot{q_1}} \times o_4 - o_1$$

Joint velocity contribution

Linear velocity due to rotation

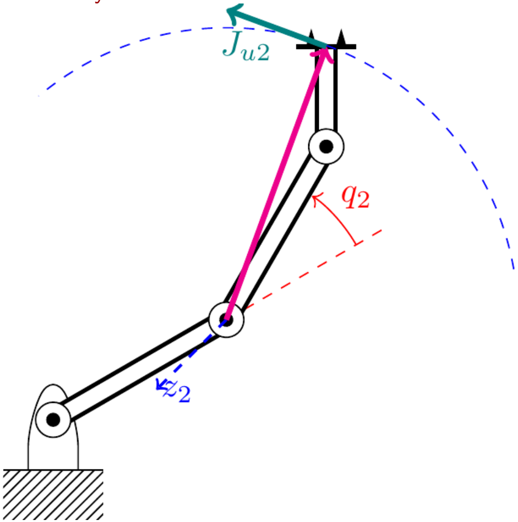


$$\vec{u_1} = \vec{\dot{q_1}} \times o_4 - o_1$$

$$\vec{u_1} = (\vec{z_1} \times o_4 - o_1)\dot{q}_1 = \vec{J_{u1}}\dot{q}_1$$

Joint velocity contribution

Linear velocity due to rotation



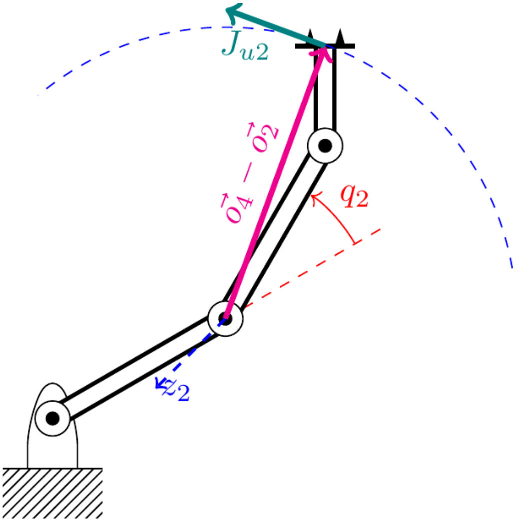
$$\vec{u_1} = \vec{\dot{q_1}} \times o_4 - o_1$$

$$\vec{u_1} = (\vec{z_1} \times o_4 - o_1)\dot{q_1} = \vec{J_{u1}}\dot{q_1}$$

$$\vec{u_2} = \vec{\dot{q_2}} \times o_4 - o_2$$

Joint velocity contribution

Linear velocity due to rotation



$$\vec{u_1} = \vec{\dot{q_1}} \times o_4 - o_1$$

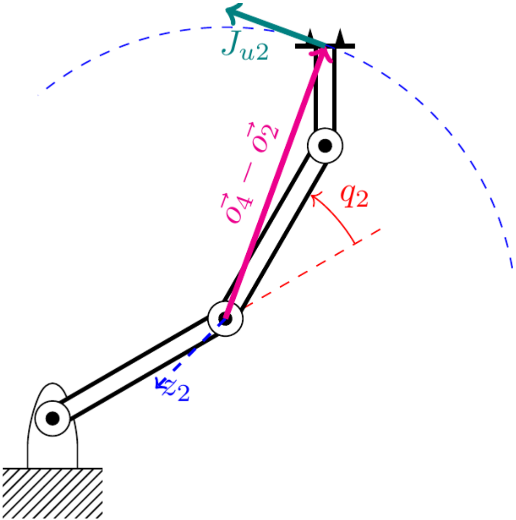
$$\vec{u_1} = (\vec{z_1} \times o_4 - o_1)\dot{q_1} = \vec{J_{u1}}\dot{q_1}$$

$$\vec{u_2} = \vec{\dot{q_2}} \times o_4 - o_2$$

$$\vec{u_2} = (\vec{z_2} \times o_4 - o_2)\dot{q_2} = \vec{J_{u2}}\dot{q_2}$$

Joint velocity contribution

Linear velocity due to rotation



$$\vec{u_1} = \vec{\dot{q_1}} \times o_4 - o_1$$

$$\vec{u_1} = (\vec{z_1} \times o_4 - o_1)\dot{q_1} = \vec{J_{u1}}\dot{q_1}$$

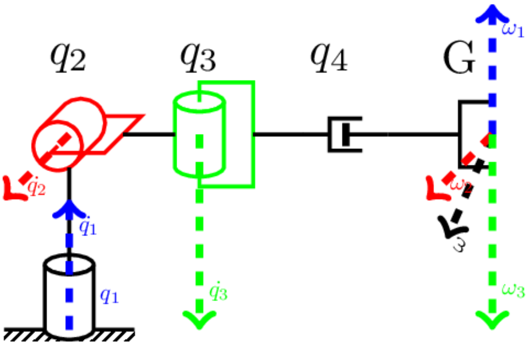
$$\vec{u_2} = \vec{\dot{q_2}} \times o_4 - o_2$$

$$\vec{u_2} = (\vec{z_2} \times o_4 - o_2)\dot{q_2} = \vec{J_{u2}}\dot{q_2}$$

$$\vec{u_i} = (\vec{z_i} \times o_n - o_i)\dot{q_i} = \vec{J_{ui}}\dot{q_i}$$

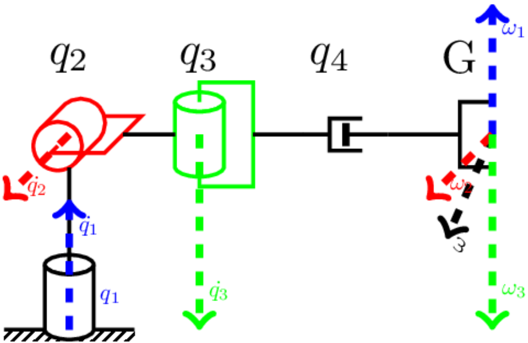
Joint velocity contribution

Angular velocity due to rotation



Joint velocity contribution

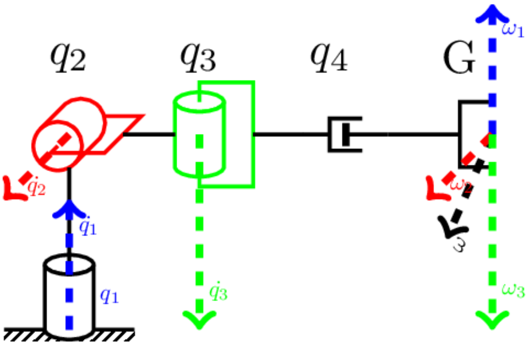
Angular velocity due to rotation



$$\vec{\omega_1} = \vec{\dot{q_1}}$$

Joint velocity contribution

Angular velocity due to rotation

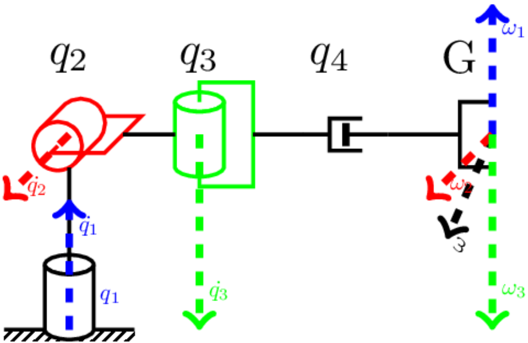


$$\vec{\omega_1} = \vec{\dot{q_1}}$$

$$\vec{\omega_1} = \vec{z_1}\dot{q}_1 = \vec{J_{\omega 1}}\dot{q}_1$$

Joint velocity contribution

Angular velocity due to rotation



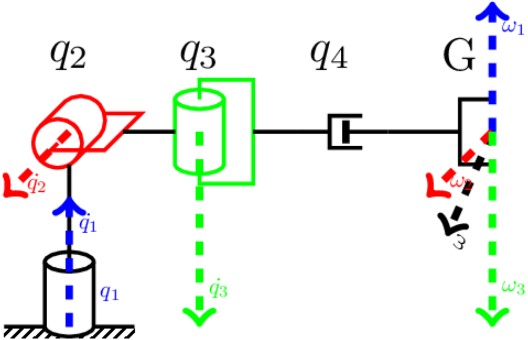
$$\vec{\omega_1} = \vec{\dot{q_1}}$$

$$\vec{\omega_1} = \vec{z_1}\dot{q_1} = \vec{J_{\omega 1}}\dot{q_1}$$

$$\vec{\omega_2} = \vec{z_2}\dot{q_2} = \vec{J_{\omega 2}}\dot{q_2}$$

Joint velocity contribution

Angular velocity due to rotation



$$\vec{\omega_1} = \vec{\dot{q_1}}$$

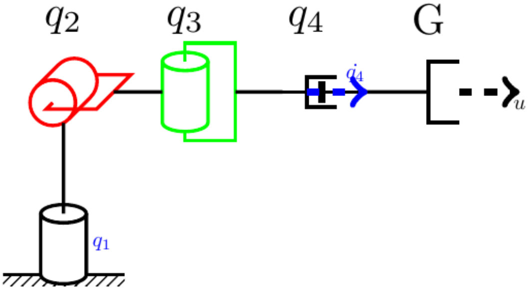
$$\vec{\omega_1} = \vec{\dot{z_1}} \dot{q_1} = \vec{J_{\omega 1}} \dot{q_1}$$

$$\vec{\omega_2} = \vec{\dot{z_2}} \dot{q_2} = \vec{J_{\omega 2}} \dot{q_2}$$

$$\vec{\omega_i} = \vec{\dot{z_i}} \dot{q_i} = \vec{J_{\omega i}} \dot{q_i}$$

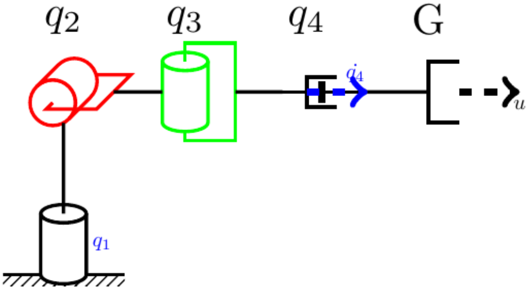
Joint velocity contribution

Linear velocity due to translation



Joint velocity contribution

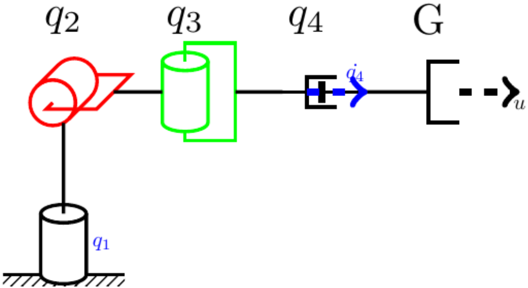
Linear velocity due to translation



$$\vec{u_1} = \vec{\dot{q_4}}$$

Joint velocity contribution

Linear velocity due to translation

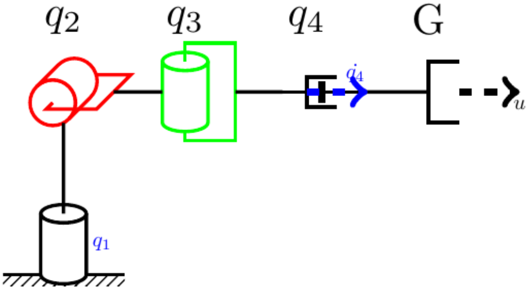


$$\vec{u_1} = \vec{\dot{q}_4}$$

$$\vec{u_1} = \vec{z_4}\dot{q}_4 = \vec{J_{u4}}\dot{q}_4$$

Joint velocity contribution

Linear velocity due to translation



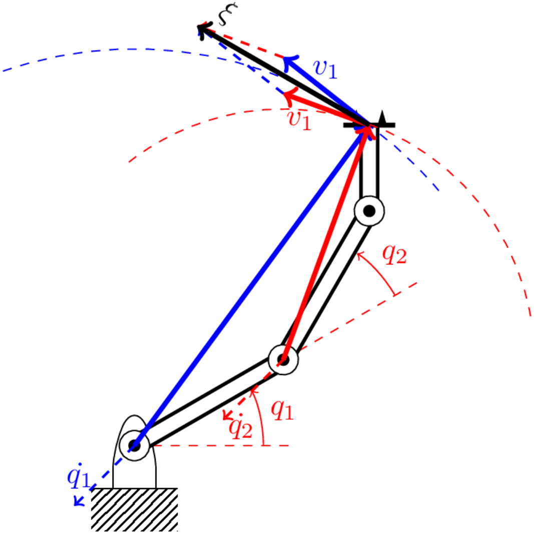
$$\vec{u_1} = \vec{\dot{q_4}}$$

$$\vec{u_1} = \vec{z_4}\dot{q}_4 = \vec{J_{u4}}\dot{q}_4$$

Do we have angular velocity due to translation?

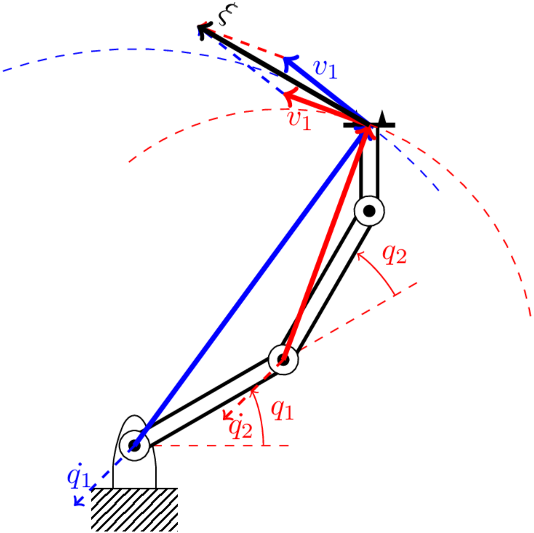
Robot velocity

Velocity addition



Robot velocity

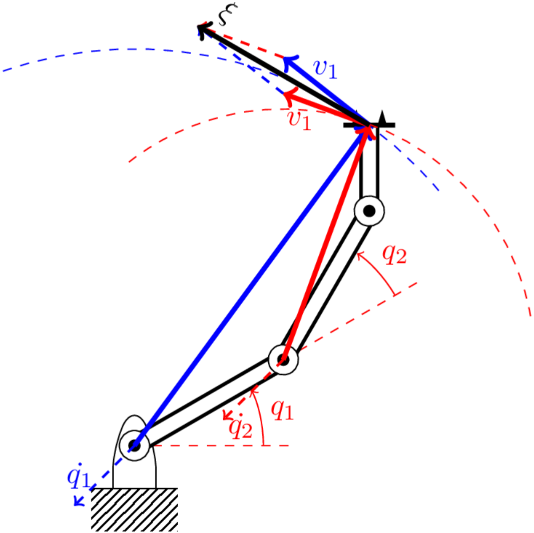
Velocity addition



$$\vec{u} = \vec{u_1} + \vec{u_2} + \cdots + \vec{u_n}$$

Robot velocity

Velocity addition

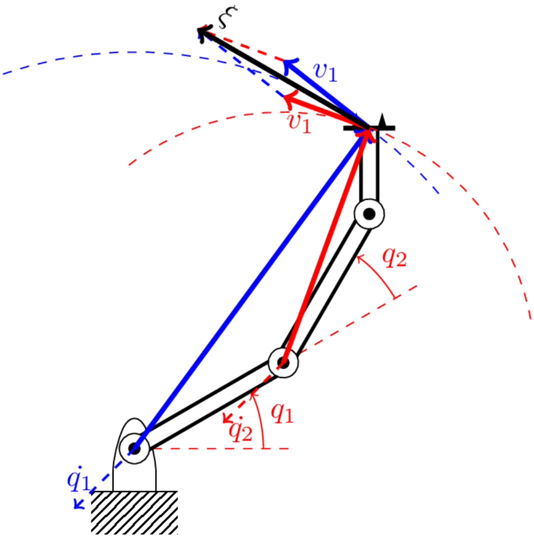


$$\vec{u} = \vec{u_1} + \vec{u_2} + \cdots + \vec{u_n}$$

$$\vec{u} = \vec{J_{u1}}\dot{q_1} + \vec{J_{u2}}\dot{q_2} + \cdots + \vec{J_{un}}\dot{q_n}$$

Robot velocity

Velocity addition



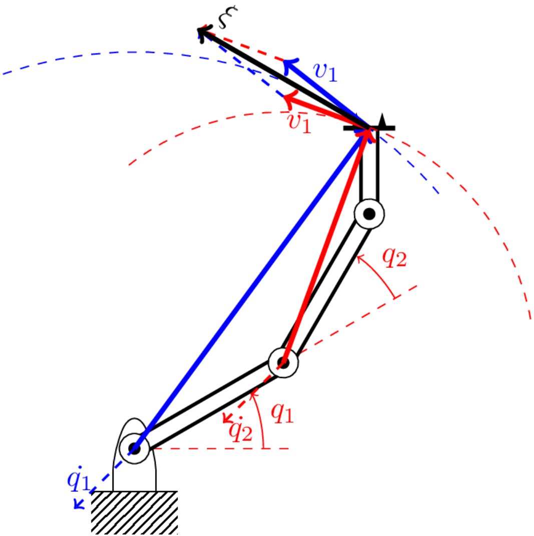
$$\vec{u} = \vec{u_1} + \vec{u_2} + \cdots + \vec{u_n}$$

$$\vec{u} = \vec{J_{u1}}\dot{q_1} + \vec{J_{u2}}\dot{q_2} + \cdots + \vec{J_{un}}\dot{q_n}$$

$$\vec{u} = \begin{bmatrix} \vec{J_{u1}} & \vec{J_{u2}} & \ldots & \vec{J_{un}} \end{bmatrix} \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \\ \vdots \\ \dot{q_n} \end{bmatrix} = J_u \dot{q}$$

Robot velocity

Velocity addition



$$\vec{u} = \vec{u_1} + \vec{u_2} + \cdots + \vec{u_n}$$

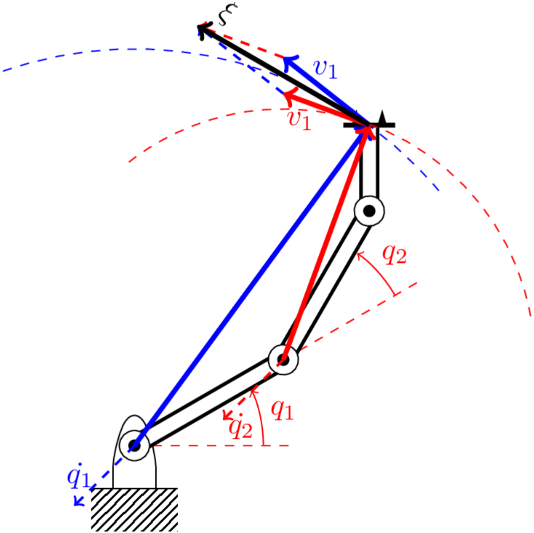
$$\vec{u} = \vec{J_{u1}}\dot{q_1} + \vec{J_{u2}}\dot{q_2} + \cdots + \vec{J_{un}}\dot{q_n}$$

$$\vec{u} = \begin{bmatrix} \vec{J_{u1}} & \vec{J_{u2}} & \dots & \vec{J_{un}} \end{bmatrix} \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \\ \vdots \\ \dot{q_n} \end{bmatrix} = J_u \dot{q}$$

$$\vec{\omega} = \begin{bmatrix} \vec{J_{\omega 1}} & \vec{J_{\omega 2}} & \dots & \vec{J_{\omega n}} \end{bmatrix} \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \\ \vdots \\ \dot{q_n} \end{bmatrix} = J_\omega \dot{q}$$

Robot velocity

Velocity addition

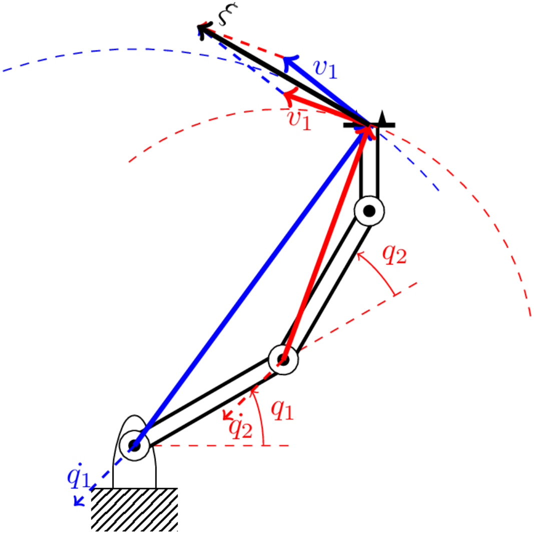


$$\vec{u} = \begin{bmatrix} \vec{J_{u1}} & \vec{J_{u2}} & \dots & \vec{J_{un}} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\vec{\omega} = \begin{bmatrix} \vec{J_{\omega 1}} & \vec{J_{\omega 2}} & \dots & \vec{J_{\omega n}} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Robot velocity

Velocity addition



$$\vec{u} = \begin{bmatrix} \vec{J_{u1}} & \vec{J_{u2}} & \dots & \vec{J_{un}} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\vec{\omega} = \begin{bmatrix} \vec{J_{\omega 1}} & \vec{J_{\omega 2}} & \dots & \vec{J_{\omega n}} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\xi = \begin{bmatrix} \vec{u} \\ \vec{\omega} \end{bmatrix} = \begin{bmatrix} J_u \\ J_\omega \end{bmatrix} \dot{q} = J\dot{q}$$

Robot velocity

Defining the Jacobian

We define a matrix called the 'Jacobian' that shows us how can we calculate the end-effector velocity if we know the joint velocities

$$\xi = J\dot{q}$$

Robot velocity

The Jacobian

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \xi = J\dot{q} = J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

By vector ξ we denote a vector that contains 6 velocities, 3 linear and 3 angular. By vector $\dot{q}$ we denote a vector containing all the n joint velocities.

Robot velocity

The Jacobian

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \xi = J\dot{q} = J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

By vector ξ we denote a vector that contains 6 velocities, 3 linear and 3 angular. By vector $\dot{q}$ we denote a vector containing all the n joint velocities.

What is the size of the Jacobian matrix J?

Defining the Jacobian

Combining angular and linear velocities

We can calculate each column of the Jacobian matrix individually. Each column represents one joint. If joint i is revolute, then:

$$J_{ir} = \begin{bmatrix} z_i \times (o_{n+1} - o_i) \\ z_i \end{bmatrix}$$

If joint i is prismatic, then:

$$J_{ip} = \begin{bmatrix} z_i \\ 0 \end{bmatrix}$$

Defining the Jacobian

Combining angular and linear velocities

$$J_{ir} = \begin{bmatrix} z_i \times (o_{n+1} - o_i) \\ z_i \end{bmatrix}, J_{ip} = \begin{bmatrix} z_i \\ 0 \end{bmatrix}$$

What is z_i?

Defining the Jacobian

Combining angular and linear velocities

$$J_{ir} = \begin{bmatrix} z_i \times (o_{n+1} - o_i) \\ z_i \end{bmatrix}, J_{ip} = \begin{bmatrix} z_i \\ 0 \end{bmatrix}$$

What is z_i?

$$R_0^i = \begin{bmatrix} X_x & Y_x & Z_x & P_x \\ X_y & Y_y & Z_y & P_y \\ X_z & Y_z & Z_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Defining the Jacobian

Combining angular and linear velocities

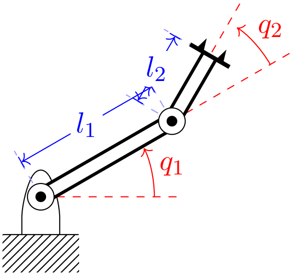
$$J_{ir} = \begin{bmatrix} z_i \times (o_{n+1} - o_i) \\ z_i \end{bmatrix}, J_{ip} = \begin{bmatrix} z_i \\ 0 \end{bmatrix}$$

What is z_i?
What is o_i

$$R_0^i = \begin{bmatrix} X_x & Y_x & Z_x & P_x \\ X_y & Y_y & Z_y & P_y \\ X_z & Y_z & Z_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Defining the Jacobian

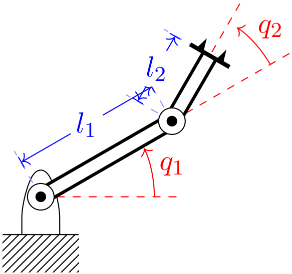
Example in $\mathbb{R}^2$



$$R_0^1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Defining the Jacobian

Example in $\mathbb{R}^2$

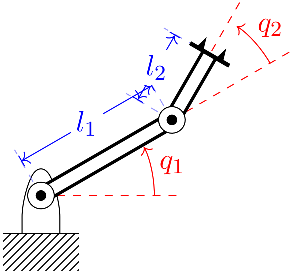


$$R_0^1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_0^2 = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Defining the Jacobian

Example in $\mathbb{R}^2$



$$R_0^1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_0^2 = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_0^3 = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Defining the Jacobian

2 link planar manipulator

$$R_0^1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_0^2 = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_0^3 = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} z_1 \times (o_3 - o_1) & z_2 \times (o_3 - o_2) \\ z_1 & z_2 \end{bmatrix}$$

Defining the Jacobian

2 link planar manipulator

$$R_0^1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_0^2 = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_0^3 = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

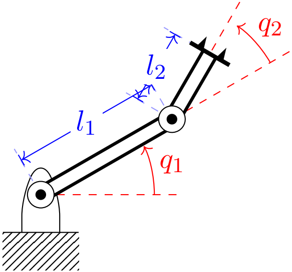
$$J = \begin{bmatrix} z_1 \times (o_3 - o_1) & z_2 \times (o_3 - o_2) \\ z_1 & z_2 \end{bmatrix}$$

where:

$$o_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, o_2 = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}, o_3 = \begin{bmatrix} l_1 c_1 + l_2 c_{1,2} \\ l_1 s_1 + l_2 s_{1,2} \\ 0 \end{bmatrix}, z_1 = z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Defining the Jacobian

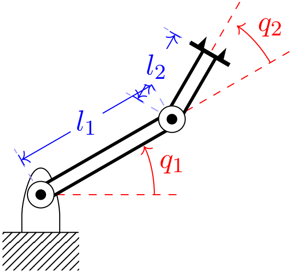
2 link planar manipulator



$$J = \begin{bmatrix} -l_1 s_1 - l_2 s_{1,2} & -l_2 s_{1,2} \\ l_1 c_1 + l_2 c_{1,2} & l_2 c_{1,2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Defining the Jacobian

2 link planar manipulator

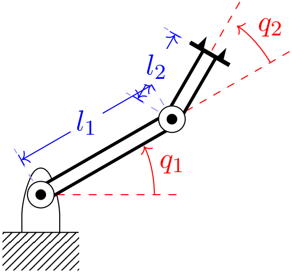


$$J = \begin{bmatrix} -l_1 s_1 - l_2 s_{1,2} & -l_2 s_{1,2} \\ l_1 c_1 + l_2 c_{1,2} & l_2 c_{1,2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

The Jacobian is a function of joint coordinates!

Defining the Jacobian

2 link planar manipulator



$$J = \begin{bmatrix} -l_1 s_1 - l_2 s_{1,2} & -l_2 s_{1,2} \\ l_1 c_1 + l_2 c_{1,2} & l_2 c_{1,2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

How do we 'use' the Jacobian?

Jacobian

Inverse velocity

We now have a method to define the end-effector velocity (angular and linear) based on the joint velocities

$$\xi = J\dot{q}$$

Jacobian

Inverse velocity

We now have a method to define the end-effector velocity (angular and linear) based on the joint velocities

$$\xi = J\dot{q}$$

Is this useful?

Jacobian

Inverse velocity

We now have a method to define the end-effector velocity (angular and linear) based on the joint velocities

$$\xi = J\dot{q}$$

Is this useful?

How do we do the opposite (i.e. define the joint velocities for specific end-effector velocity)?

Jacobian

Inverse velocity

We now have a method to define the end-effector velocity (angular and linear) based on the joint velocities

$$\xi = J\dot{q}$$

Is this useful?

How do we do the opposite (i.e. define the joint velocities for specific end-effector velocity)?

$$J^{-1}\xi = \dot{q}$$

Jacobian

Inverting the velocity

Is J always inversible?

Is J always inversible?

Conditions for Jacobian inversibility

- The Jacobian must be square
- The rank of the Jacobian must be equal to its size

Jacobian

Inverting the velocity

Is J always inversible?

Conditions for Jacobian inversibility

- The Jacobian must be square
- The rank of the Jacobian must be equal to its size

For achieving any velocity in $\mathbb{R}^3$, the Jacobian must be 6×6. What do we need for such a Jacobian?

Jacobian

The pseudoinverse

In the cases we cannot invert the Jacobian (e.g. we don't have 6 joints), we can calculate the *pseudoinverse*.

Jacobian

The pseudoinverse

In the cases we cannot invert the Jacobian (e.g. we don't have 6 joints), we can calculate the *pseudoinverse*.

$$\textit{For } J \in \mathbb{R}^{m\times n}, \textit{ if } m < n, \textit{ then } (JJ^T)^{-1} exists.$$

Jacobian

The pseudoinverse

In the cases we cannot invert the Jacobian (e.g. we don't have 6 joints), we can calculate the *pseudoinverse*.

$$\textit{For } J \in \mathbb{R}^{m\times n}, \textit{ if } m < n, \textit{ then } (JJ^T)^{-1} exists.$$

$$(JJ^T)(JJ^T)^{-1} = I$$
$$J[J^T(JJ^T)^{-1}] = I$$
$$JJ^+ = I$$

where:

$$J^+ = J^T(JJ^T)^{-1}$$

Jacobian

The pseudoinverse

In the cases we cannot invert the Jacobian (e.g. we don't have 6 joints), we can calculate the *pseudoinverse*.

$$\textit{For } J \in \mathbb{R}^{m \times n}, \textit{ if } m < n, \textit{ then } (JJ^T)^{-1} exists.$$

$$(JJ^T)(JJ^T)^{-1} = I$$
$$J[J^T(JJ^T)^{-1}] = I$$
$$JJ^+ = I$$

where:

$$J^+ = J^T(JJ^T)^{-1}$$

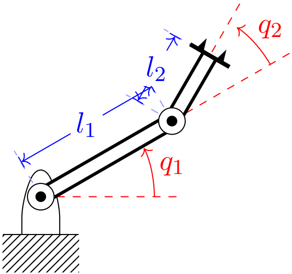
therefore:

$$\dot{q} = J^+\xi$$

Jacobian

Controlling specific velocities

When we have less than 6 joints, we can also choose to control only specific velocities of the end-effector

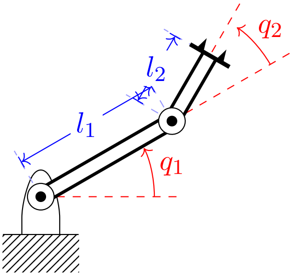


$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = J_u \dot{q} = \begin{bmatrix} 0 & 0 \\ -l_1 s_1 - l_2 s_{1,2} & -l_2 s_{1,2} \\ l_1 c_1 + l_2 c_{1,2} & l_2 c_{1,2} \end{bmatrix} \dot{q}$$

Jacobian

Controlling specific velocities

When we have less than 6 joints, we can also choose to control only specific velocities of the end-effector

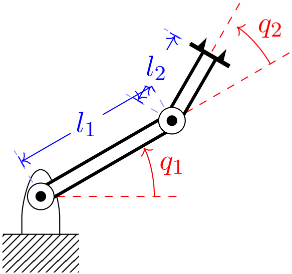


$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = J_u\dot{q} = \begin{bmatrix}0 & 0\\ -l_1s_1 - l_2s_{1,2} & -l_2s_{1,2}\\ l_1c_1 + l_2c_{1,2} & l_2c_{1,2}\end{bmatrix}\dot{q}$$

$$\begin{bmatrix}\dot{y}\\ \dot{z}\end{bmatrix} = J_{u,red}\dot{q} = \begin{bmatrix}-l_1s_1 - l_2s_{1,2} & -l_2s_{1,2}\\ l_1c_1 + l_2c_{1,2} & l_2c_{1,2}\end{bmatrix}\dot{q}$$

Jacobian inverse

2 link planar manipulator



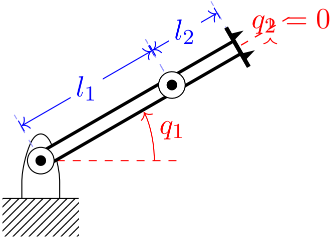
$$J_u = \begin{bmatrix} -l_1 s_1 - l_2 s_{1,2} & -l_2 s_{1,2} \\ l_1 c_1 + l_2 c_{1,2} & l_2 c_{1,2} \end{bmatrix}$$

$$J_u^{-1} = \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 c_{1,2} & l_2 s_{1,2} \\ -l_1 c_1 - l_2 c_{1,2} & -l_1 s_1 - l_2 s_{1,2} \end{bmatrix}$$

Jacobian inverse

2 link planar manipulator

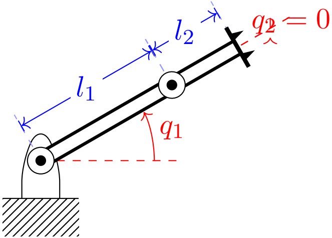
What happens when $q_2 = 0$?



Jacobian inverse

2 link planar manipulator

What happens when $q_2 = 0$?



$$J_u^{-1} = \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 c_{1,2} & l_2 s_{1,2} \\ -l_1 c_1 - l_2 c_{1,2} & -l_1 s_1 - l_2 s_{1,2} \end{bmatrix}$$

Jacobian

Singularities

The Jacobian is a function of the joint coordinates q, and therefore it varies for different robot configurations.

The Jacobian is a function of the joint coordinates q, and therefore it varies for different robot configurations.

In some cases, the Jacobian might lose rank, or might become non-invertible, (i.e. determinant equal to zero)

In such cases, the robot loses dexterity, or even a degree of freedom.

Robot manipulability

Why does this all matter?

The Jacobian allows us to map joint velocities to end-effector velocities. We have seen that at different configurations, we have a different map (since J depends on q).

Robot manipulability

Why does this all matter?

The Jacobian allows us to map joint velocities to end-effector velocities. We have seen that at different configurations, we have a different map (since J depends on q).

Can we quantify how much dexterity our robot has at different configurations? (i.e. manipulability?)

Robot manipulability

Why does this all matter?

The Jacobian allows us to map joint velocities to end-effector velocities. We have seen that at different configurations, we have a different map (since J depends on q).

Can we quantify how much dexterity our robot has at different configurations? (i.e. manipulability?)

hint: yes!

We model our robot as an input-output system (input is joint velocities, output is end-effector velocities). If we consider unit inputs, then we have:

$$q^T q = 1$$

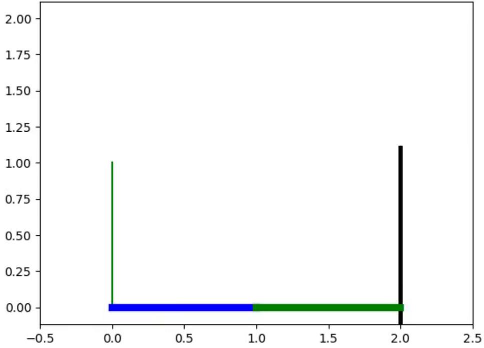
which we can write as:

$$\xi^T (J J^T)^{-1} \xi = 1$$

which is the equation of an $m - dimensional$ ellipsoid.

Robot manipulability

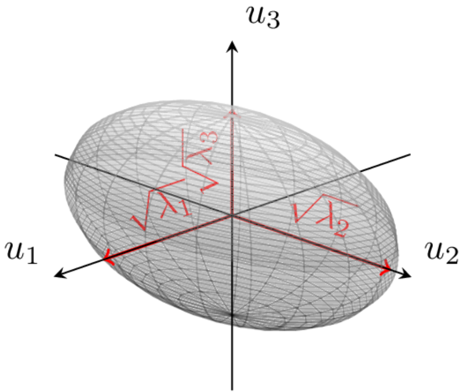
Velocity ellipse



In 2D, we have a 2-dimensional ellipsoid, i.e. an ellipse.

Robot manipulability

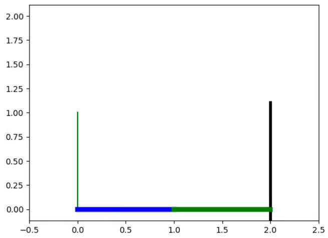
Velocity ellipse



In 3D, we have a 3-dimensional ellipsoid

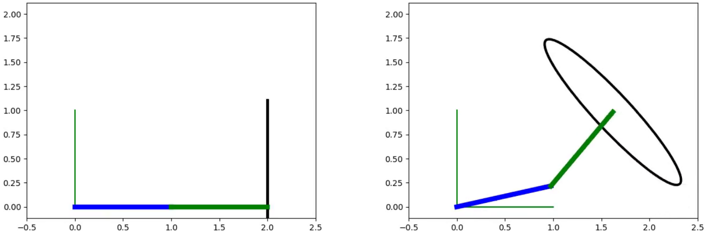
Robot manipulability

Velocity ellipse



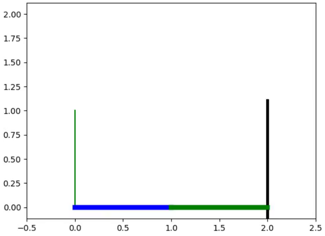
Robot manipulability

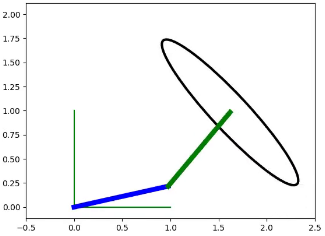
Velocity ellipse

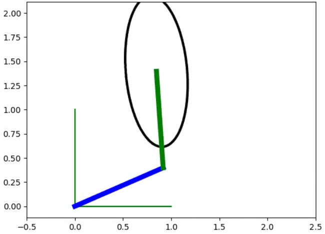


Robot manipulability

Velocity ellipse

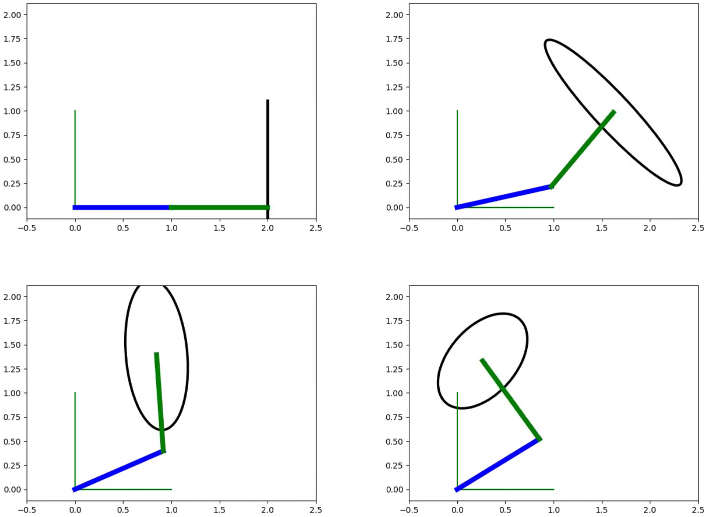






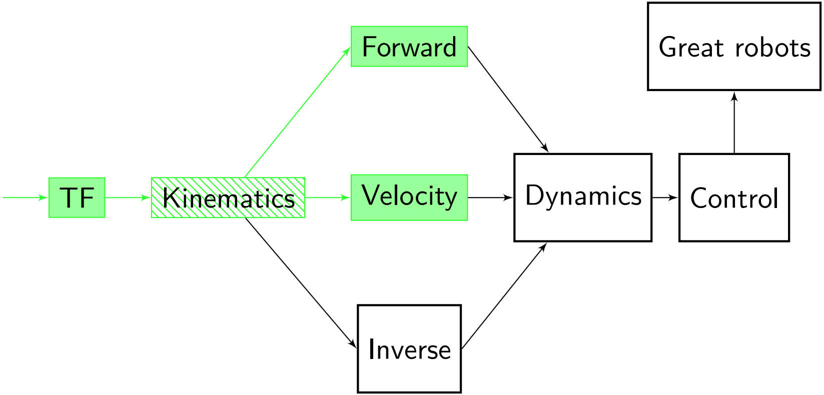
Robot manipulability

Velocity ellipse



Grand scheme

The big picture



Questions?