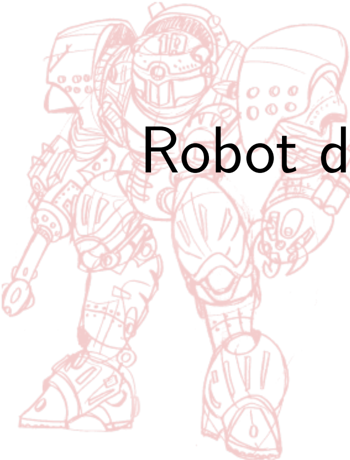

Robot dynamic modeling

TECHNICAL UNIVERSITY OF CLUJ-NAPOCA ROMANIA

October 14, 2018

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Agenda

- Lagrangian of a robot
- How it all fits together

What did we do last week?
Recap

Lagrange defined a basic quatity for any system of bodies as the difference between its kinetic and potential energy.

$$L = K - P$$

We call this quantity the Lagrangian of the system.

What did we do last week?

Recap

We define the Lagrangian as the difference between Kinetic and Potential energy of our system

$$L = K - P$$

where:

Potential Energy

$$P = mgh$$

Kinetic Energy

$$K = \frac{1}{2}(mu^2 + \omega^T I \omega)$$

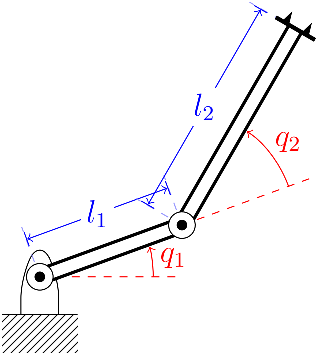
Moment of inertia

$$I = 3 \times 3$$

Lagrangian of a robot

How do we calculate it?

Let's take an 'easy' example of a 2-link planar robot.



Let's assume that segments have masses m_1 and m_2 respectively.

Lagrangian of a robot

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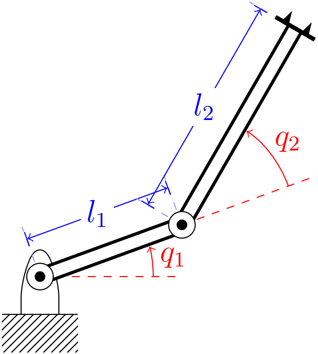


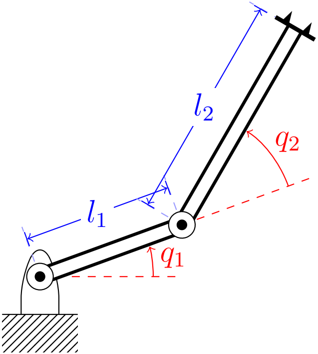
Let's assume that segments have masses m_1 and m_2 respectively.

We need to calculate its Lagrangian

Lagrangian of a robot

Dynamic energy

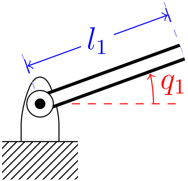
We need to calculate the total dynamic energy of the system.



The total dynamic energy is the sum of the dynamic energies of each segment. What is the dynamic energy of each segment?

Lagrangian of a robot

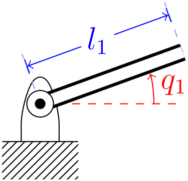
Dynamic energy



We consider the mass of the link to be concentrated at its center of mass.

Lagrangian of a robot

Dynamic energy

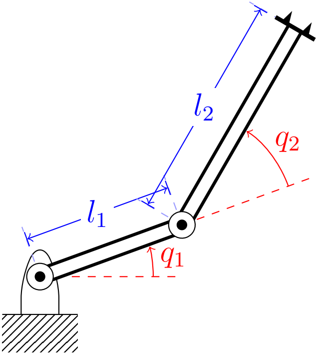


We consider the mass of the link to be concentrated at its center of mass. Therefore:

$$P_1 = m_1 g \frac{l_1}{2} sin q_1$$

Lagrangian of a robot

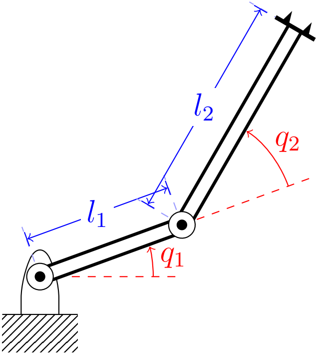
Dynamic energy



For the second segment, we think alike:

Lagrangian of a robot

Dynamic energy

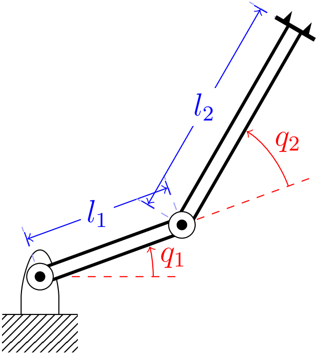


For the second segment, we think alike:

$$P_2 = m_2 g \left(l_1 sinq_1 + \frac{l_2}{2} sinq_2 \right)$$

Lagrangian of a robot

Dynamic energy



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The total potential energy is therefore:

$$P = m_1 g \frac{l_1}{2} sinq_1 + m_2 g \left(l_1 sinq_1 + \frac{l_2}{2} sinq_2 \right)$$

Lagrangian of a robot

Kinetic energy

Once again, we take the kinetic energy of each segment and add them together.

$K_{total} = K_1 + K_2$

Lagrangian of a robot

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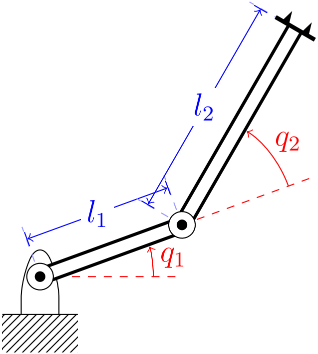
Let's start with the linear kinetic energy first.



Lagrangian of a robot

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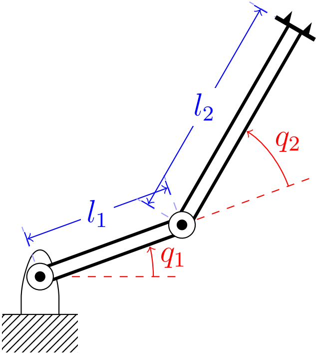

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Lagrangian of a robot

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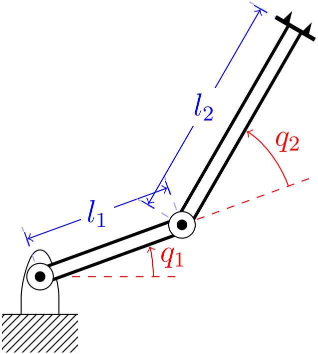

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Lagrangian of a robot

Going back in time

How do we convert the joint velocity $(\dot{q})$ into linear velocity (u)?

Lagrangian of a robot

Going back in time

How do we convert the joint velocity ($\dot{q}$) into linear velocity (u)?

The jacobian!

$$u = J_u \dot{q}$$

Lagrangian of a robot

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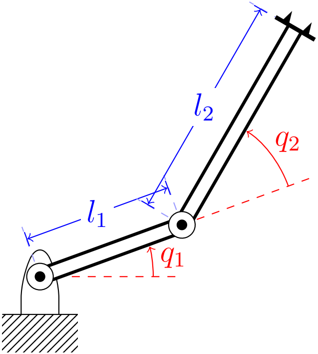
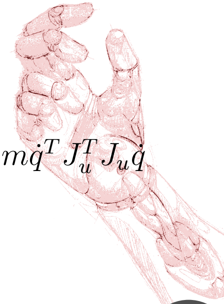


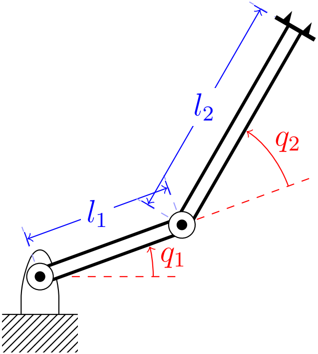
$$K_{lin} = \frac{1}{2}mu^2 = \frac{1}{2}m\dot{q}^T J_u^T J_u \dot{q}$$



Lagrangian of a robot

Kinetic energy

... and then the angular kinetic energy:

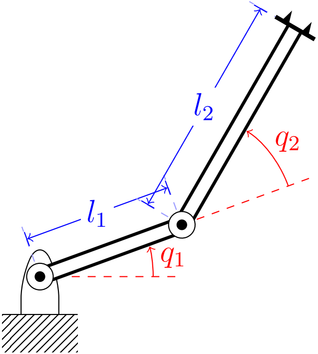


$$K_{lin} = \frac{1}{2} I \omega^2 = \frac{1}{2} \dot{q}^T J_\omega^T R I R^T J_\omega \dot{q}$$

Lagrangian of a robot

Kinetic energy

Therefore, the total Kinetic energy of the robot is:



$$K = \frac{1}{2}\dot{q}^T \sum_{i=1}^{n} \left[m_i J_{vi}^T J_{vi} + J_{\omega i} R_i I_i R_i^T J_{\omega i} \right] \dot{q}$$

Lagrangian of a robot

Let's plug it in the Lagrangian equation of motion

Eventually, we can write the kinetic energy in a condensed format:

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

And the potential energy:

$$P = \sum_{i=1}^{n} g h_i m_i$$

Therefore, the total Lagrangian is:

$$L = K - P = \frac{1}{2}\dot{q}^T D(q)\dot{q} - \sum_{i=1}^{n} g h_i m_i$$

Lagrangian of a robot

Let's plug it all together

The equation of motion is:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}\dot{q}_j$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}\ddot{q}_j + \sum_j \frac{d}{dt} d_{kj}\dot{q}_j = \sum_j d_{kj}\ddot{q}_j + \sum_{i,j} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

Lagrangian of a robot

Let's plug it all together

The second term is:

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k}$$

Therefore, everything together is:

$$\sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k} = \tau_k$$

Lagrangian of a robot

Condensed form

We can write this equation in a more general form:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

Lagrangian of a robot

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The matrix D, contains information about the **inertia** of the system, therefore contains all the masses and moments of inertia.

Lagrangian of a robot

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Lagrangian of a robot

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Finally, the term g contains the dependence of the potential energy from the position of the robot.

Lagrangian of a robot

Christoffel symbols

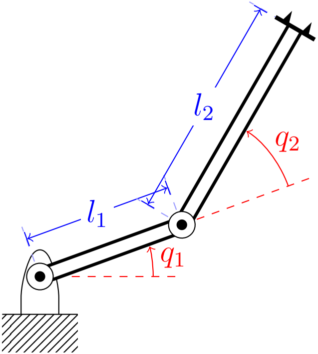
The k, j-th element of matrix $C(q, \dot{q})$ is defined as:

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$$

$$= \sum_{i=1}^{n} \frac{1}{2}\{\frac{\partial d_{kj}}{\partial q_j} + \frac{\partial_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\}\dot{q}_i$$

Lagrangian of a robot

Let's apply it on this robot



Questions?