Robot dynamic modeling

TECHNICAL UNIVERSITY OF CLUJ-NAPOCA ROMANIA

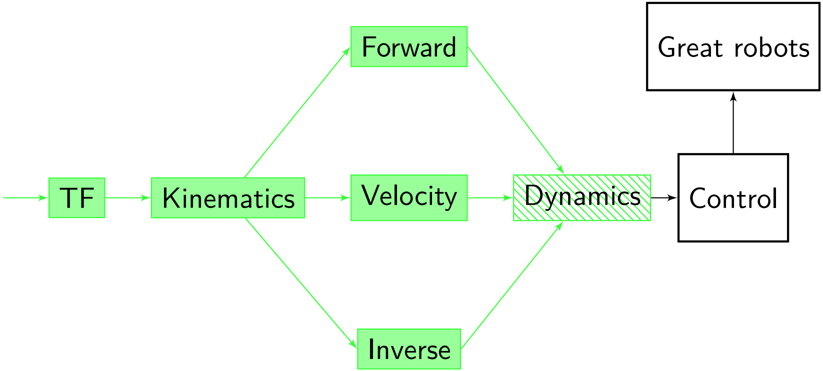
Last update: November 19, 2024

Agenda

- Lagrangian of a robot
- How it all fits together

Grand scheme

The big picture



What did we do last week?

Recap

Lagrange defined a basic quantity for any system of bodies as the difference between its kinetic and potential energy.

$$L = T - V$$

We call this quantity the Lagrangian of the system.

What did we do last week?

Recap

We define the Lagrangian as the difference between Kinetic and Potential energy of our system

$$L = T - V$$

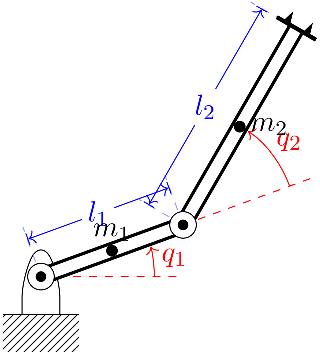
where:

Potential Energy	Kinetic Energy	Moments of inertia
$V = mgh$	$T = \frac{1}{2}(\vec{u}^T m \vec{u} + \vec{\omega}^T I \vec{\omega})$	$I = 3 \times 3$

Lagrangian of a robot

How do we calculate it?

Let's take an 'easy' example of a 2-link planar robot.



Let's assume that segments have masses m_1 and m_2 respectively.

Lagrangian of a robot

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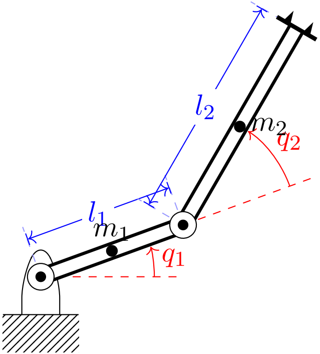

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Lagrangian of a robot

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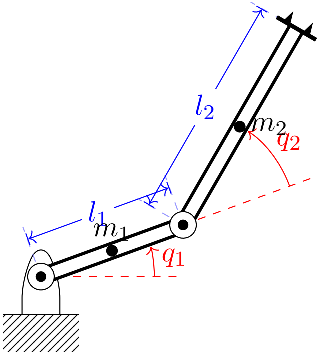


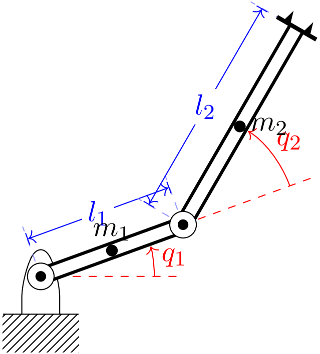
Let's assume that segments have masses m_1 and m_2 respectively.

We need to calculate its Lagrangian
in terms of some 'generalized' coordinates.

Lagrangian of a robot

How do we calculate it?

Which 'generalized' coordinates are most convenient?



Lagrangian of a robot

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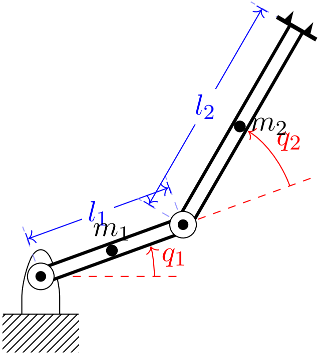

X, Y?

Lagrangian of a robot

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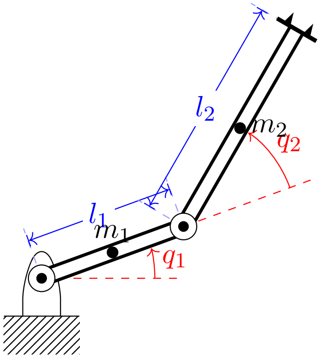

X, Y?

θ, r?

Lagrangian of a robot

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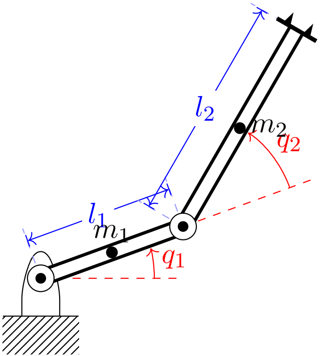

X, Y?
θ, r?
$q1$, $q2$?

Lagrangian of a robot

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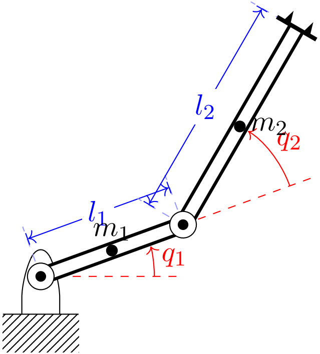
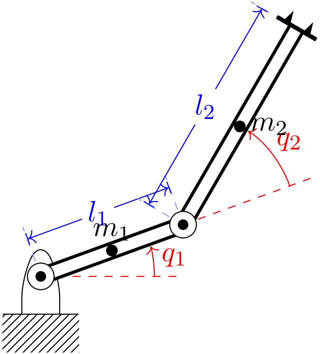


X, Y?
θ, r?
$q1$, $q2$?
Why?

Lagrangian of a robot

Potential energy

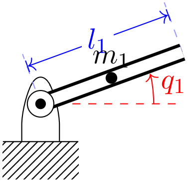
We need to calculate the total potential energy of the system with respect to $q, \dot{q}$.



The total potential energy is the sum of the potential energies of each segment. What is the potential energy of each segment?

Lagrangian of a robot

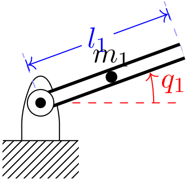
Potential energy



We consider the mass of the link to be concentrated at its center of mass.

Lagrangian of a robot

Potential energy

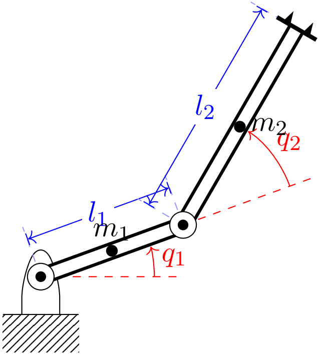


We consider the mass of the link to be concentrated at its center of mass. Therefore:

$$V_1(q, \dot{q}) = m_1 g \frac{l_1}{2} sin q_1$$

Lagrangian of a robot

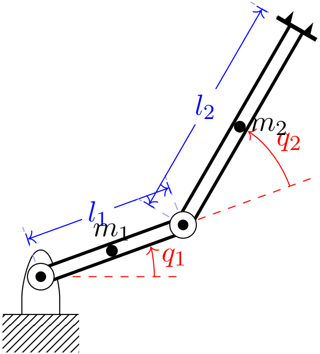
Potential energy



For the second segment, we think alike:

Lagrangian of a robot

Potential energy

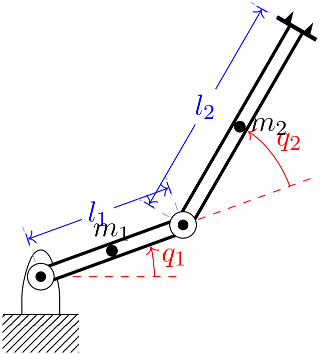


For the second segment, we think alike:

$$V_2(q,\dot{q}) = m_2 g\left(l_1 sin q_1 + \frac{l_2}{2} sin(q_1+q_2)\right)$$

Lagrangian of a robot

Potential energy



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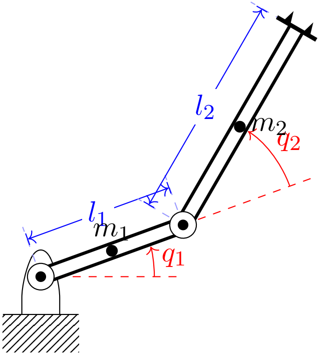
In general: $V_i = m_i g R_0^{mi}[2,3]$

The total potential energy is therefore:

$$V(q,\dot{q}) = m_1 g\frac{l_1}{2} sin q_1 + m_2 g\left(l_1 sin q_1 + \frac{l_2}{2} sin(q_1+q_2)\right) = g\sum_{i=1}^{n} m_i R_0^{mi}[2,3]$$

Lagrangian of a robot

Potential energy



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Does V depend on $\dot{q}$?

Lagrangian of a robot

Kinetic energy

Once again, we take the kinetic energy of each segment with respect to $q, \dot{q}$ and add them together.

$$T_{total}(q, \dot{q}) = T_1(q, \dot{q}) + T_2(q, \dot{q})$$

Lagrangian of a robot

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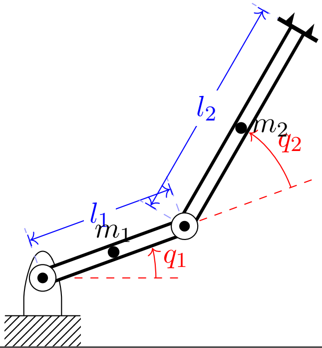
Let's start with the linear kinetic energy first.



Lagrangian of a robot

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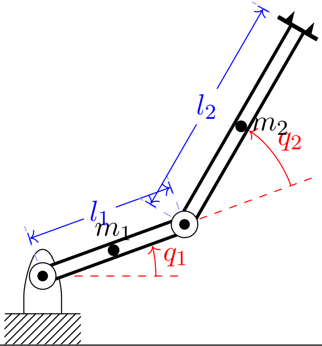

$$T_{lin}(q, \dot{q}) = \frac{1}{2} \sum_{i=1}^{n} [u_i^T m_i u_i]$$

Lagrangian of a robot
Kinetic energy

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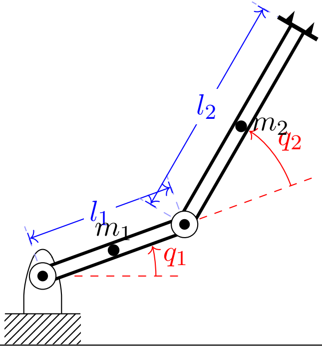


$$T_{lin}(q, \dot{q}) = \frac{1}{2}\sum_{i=1}^{n}[u_i^T m_i u_i]$$

Do we know what is u in terms of $q, \dot{q}$?

Lagrangian of a robot

Going back in time

How do we convert the linear end-effector velocity (u) into joint velocity $(\dot{q})$?

Lagrangian of a robot

Going back in time

How do we convert the linear end-effector velocity (u) into joint velocity $(\dot{q})$?

The jacobian!

$$u = J_u \dot{q}$$

Lagrangian of a robot

Going back in time

How do we convert the linear end-effector velocity (u) into joint velocity ($\dot{q}$)?

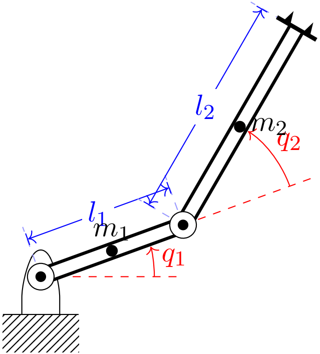
The jacobian!

$$u = J_u \dot{q}$$

But what Jacobian? What velocities do we want to calculate?

Lagrangian of a robot

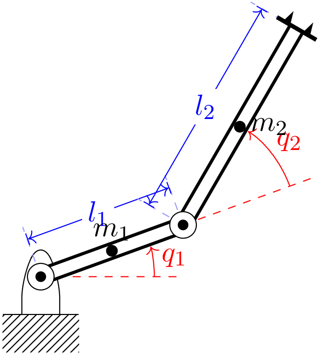
Kinetic energy



Since we need the velocity of the masses, we calculate the Jacobian *until* the masses.

Lagrangian of a robot

Kinetic energy



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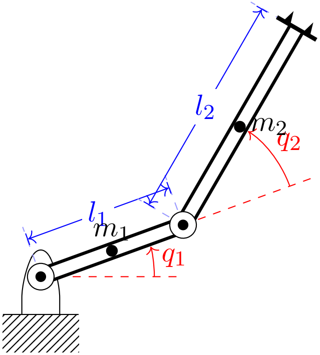
$$J_{u1} = \begin{bmatrix} z_1 \times (o_{m1} - o_1) \end{bmatrix}$$

Lagrangian of a robot

Kinetic energy



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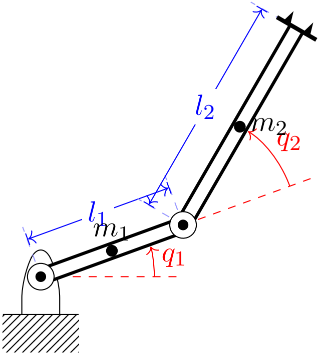
Lagrangian of a robot

Kinetic energy



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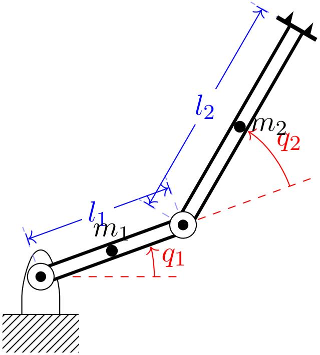
To ensure equal dimensions for the Jacobians, we *pad* the first Jacobian with zeros

Lagrangian of a robot

Kinetic energy



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Lagrangian of a robot

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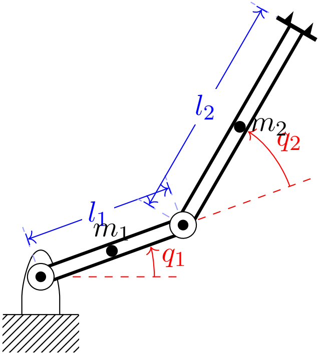

$$T_{lin}(q, \dot{q}) = \frac{1}{2}\vec{u}^T m\vec{u} = \frac{1}{2}\dot{q}^T \sum_{i=1}^{n}[J_{ui}^T m_i J_{ui}]\dot{q}$$

Lagrangian of a robot

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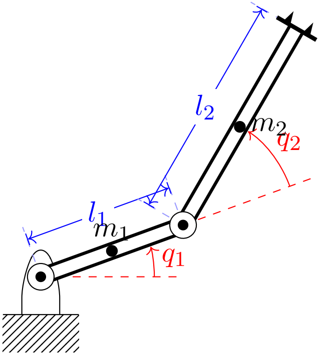

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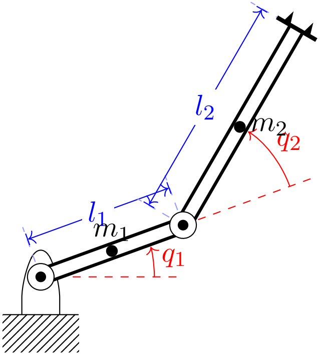
Remember $(AB)^T = B^T A^T$

Lagrangian of a robot

Kinetic energy

... and then the angular kinetic energy:



$$T_{ang}(q, \dot{q}) = \frac{1}{2} \sum_{i=1}^{n} [\omega_i^T I_i \omega_i]$$

Lagrangian of a robot

Kinetic energy

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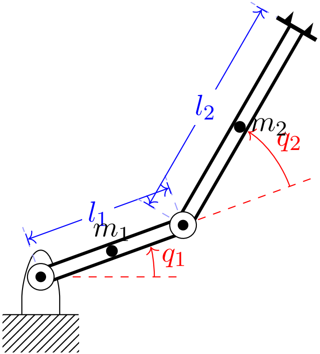

$$T_{ang}(q,\dot{q}) = \frac{1}{2}\sum_{i=1}^{n}[\omega_i^T I_i \omega_i]$$

I_i is expressed on the coordinate frame of link i

Lagrangian of a robot

Kinetic energy

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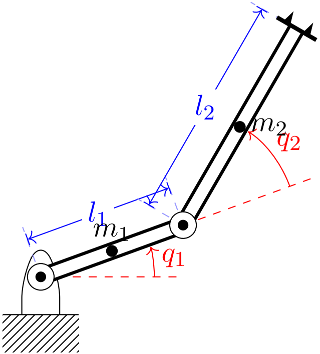

$$T_{ang}(q,\dot{q}) = \frac{1}{2}\sum_{i=1}^{n}[\omega_i^T I_i \omega_i]$$

I_i is expressed on the coordinate frame of link i
But must be 'transformed' in the base coordinate frame

Lagrangian of a robot

Kinetic energy

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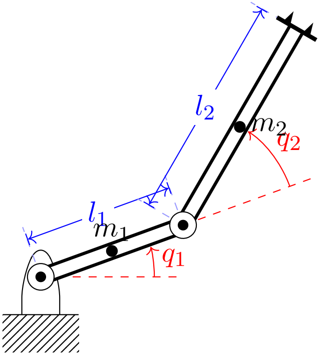

$$T_{ang}(q, \dot{q}) = \frac{1}{2} \sum_{i=1}^{n} [\omega_i^T I_i \omega_i]$$

I_i is expressed on the coordinate frame of link i
But must be 'transformed' in the base coordinate frame
$I_0^i = R_0^{m_i} I_i^i R_0^{m_i T}$

Lagrangian of a robot

Kinetic energy

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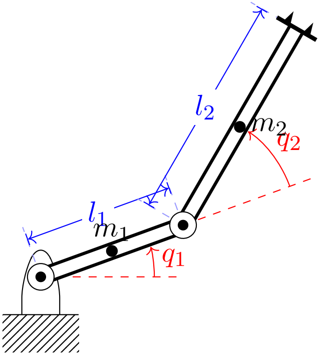

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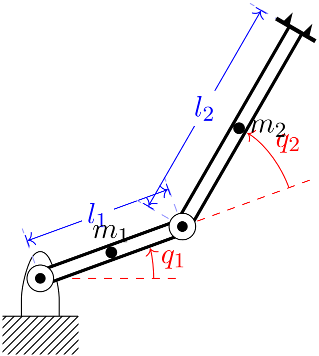
But must be 'transformed' in the base coordinate frame
$I_0^i = R_0^{m_i} I_i^i R_0^{m_i T}$
Therefore

$$T_{ang}(q, \dot{q}) = \frac{1}{2} \dot{q}^T \sum_{i=1}^{n} [J_{\omega i}^T R_0^{m_i} I_i^i R_0^{m_i T} J_{\omega i}] \dot{q}$$

Lagrangian of a robot

Kinetic energy

Therefore, the total Kinetic energy of the robot is:



$$T(q,\dot{q}) = \frac{1}{2}\dot{q}^T \sum_{i=1}^{n} \left[J_{vi}^T m_i J_{vi} + J_{\omega i}^T R_0^{m_i} I_i^i R_0^{m_i T} J_{\omega i} \right] \dot{q}$$

Lagrangian of a robot

Kinetic energy

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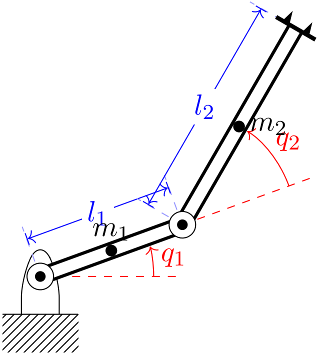

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Does T depend on $\dot{q}$?

Lagrangian of a robot

Kinetic energy

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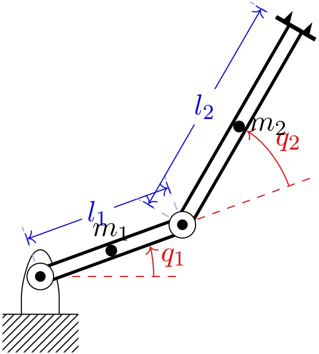

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Does T depend on $\dot{q}$? Does it depend on q?

Lagrangian of a robot

Jacobians

$$T(q,\dot{q}) = \frac{1}{2}\dot{q}^T \sum_{i=1}^{n} \left[J_{vi}^T m_i J_{vi} + J_{\omega i}^T R_0^{m_i} I_i R_0^{m_i T} J_{\omega i} \right] \dot{q}$$

What is the size of J_{ui} and $J_{\omega i}$?

Lagrangian of a robot

Jacobians

$$T(q,\dot{q}) = \frac{1}{2}\dot{q}^T \sum_{i=1}^{n} \left[J_{vi}^T m_i J_{vi} + J_{\omega i}^T R_0^{m_i} I_i R_0^{m_i T} J_{\omega i} \right] \dot{q}$$

What is the size of J_{ui} and $J_{\omega i}$?

$$J_{u1} = \begin{bmatrix} z_1 \times (o_3 - o_1) & 0 \end{bmatrix}$$

$$J_{u2} = \begin{bmatrix} z_1 \times (o_3 - o_1) & z_2 \times (o_3 - o_2) \end{bmatrix}$$

Lagrangian of a robot

Jacobians

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$$J_{\omega 1} = \begin{bmatrix} z_1 & 0 \end{bmatrix}$$

$$J_{\omega 2} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}$$

Lagrangian of a robot

Let's plug it in the Lagrangian equation of motion

Eventually, we can write the kinetic energy in condensed form:

$$T(q,\dot{q}) = \frac{1}{2}\dot{q}^T \sum_{i=1}^{n} \left[J_{vi}^T m_i J_{vi} + J_{\omega i}^T R_0^{m_i} I_i R_0^{m_i T} J_{\omega i} \right] \dot{q} = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

And the potential energy:

$$V(q) = g \sum_{i=1}^{n} m_i R_0^{m_i}[2,3]$$

Therefore, the total Lagrangian is:

$$L(q,\dot{q}) = T - V = \frac{1}{2}\dot{q}^T D(q)\dot{q} - g \sum_{i=1}^{n} m_i R_0^{m_i}[2,3]$$

Lagrangian of a robot

Let's plug it all together

If we expand the first term, we get:

$$L = T - V = \frac{1}{2}\dot{q}^T D(q)\dot{q} - g\sum_{i=1}^{n} m_i R_0^{mi}[2,3]$$

$$L = \frac{1}{2}\sum_{i,j}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j - g\sum_{i=1}^{n} m_i R_0^{mi}[2,3]$$

Lagrangian of a robot

Let's plug it all together

The equation of motion is:

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$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_k}} = \sum_j d_{kj}\ddot{q}_j + \sum_j \frac{d}{dt} d_{kj}\dot{q}_j = \sum_j d_{kj}\ddot{q}_j + \sum_{i,j} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

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The second term is:

$$\frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{i,j} \frac{\partial d_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial V}{\partial q_k}$$

Therefore, everything together is:

$$\sum_j d_{kj} \ddot{q}_j + \sum_{i,j} \{ \frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial d_{ij}}{\partial q_k} \} \dot{q}_i \dot{q}_j + \frac{\partial V}{\partial q_k} = \tau_k$$

Lagrangian of a robot

Condensed form

We can write this equation in a more general form:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau$$

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Finally, vector G contains the dependence of the potential energy from the position of the robot.

Lagrangian of a robot

Christoffel symbols

The k, j-th element of matrix $C(q, \dot{q})$ is defined as:

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$$

$$= \sum_{i=1}^{n} \frac{1}{2}\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\}\dot{q}_i$$

Inverse and forward dynamics

Again the inverse?

Inverse and forward dynamics

Again the inverse?

Which one is which?

Inverse and forward dynamics

Again the inverse?

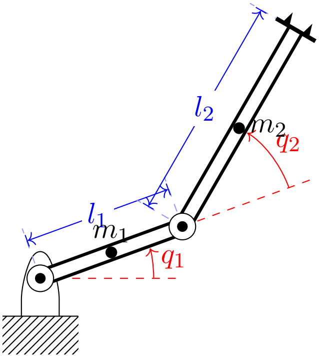
Which one is which?

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

$$D(q)^{-1}(\tau - C(q,\dot{q})\dot{q} - g(q)) = \ddot{q}$$

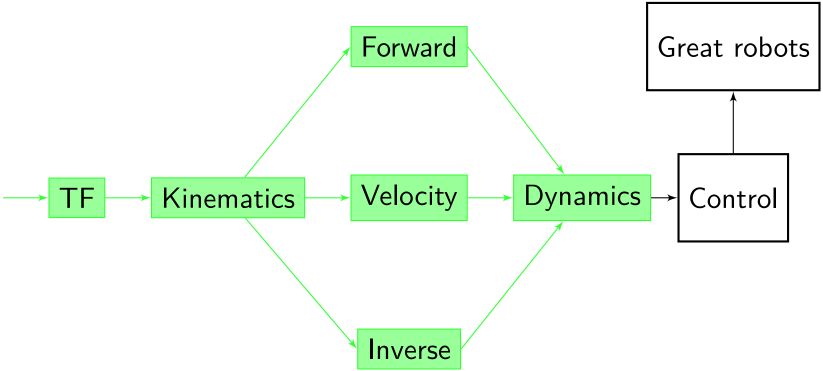
Lagrangian of a robot

Let's apply it on this robot



Grand scheme

The big picture



Questions?