Trajectories

Planning

TECHNICAL UNIVERSITY
OF CLUJ-NAPOCA
ROMANIA

November 11, 2024

Agenda

- Why trajectories?
- Interpolation
- Joint trajectories
- End-effector position trajectories
- End-effector pose trajectories

Recap

Geometric Models

Forward kinematics

I want to know where will my end-effector be, if I give specific coordinates (values) to each joint

Recap

Geometric Models

Forward kinematics

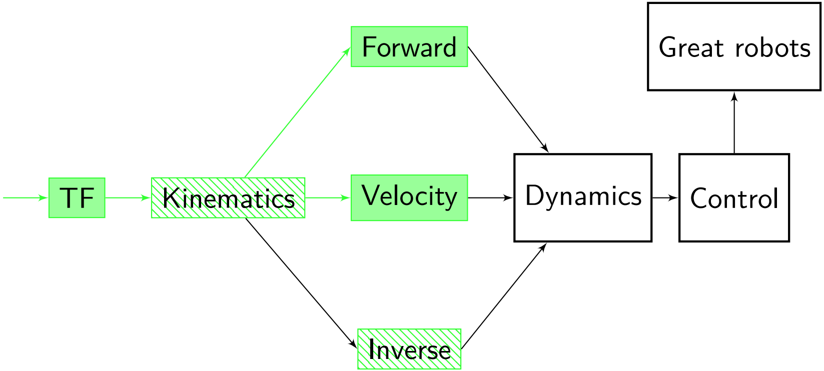
I want to know where will my end-effector be, if I give specific coordinates (values) to each joint

Inverse kinematics

I want to know what should the joint coordinates (values) be in order for my end-effector to reach a specific pose

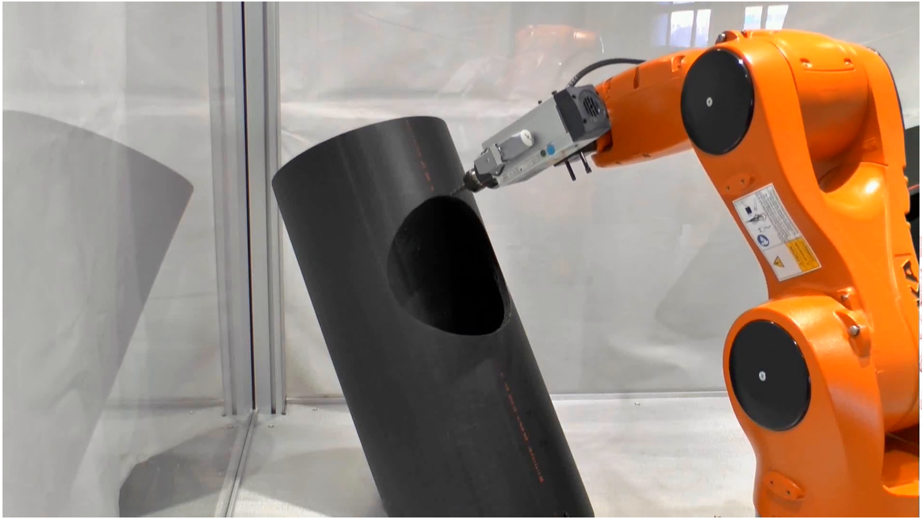
Grand scheme

The big picture



Why trajectories?

Video example

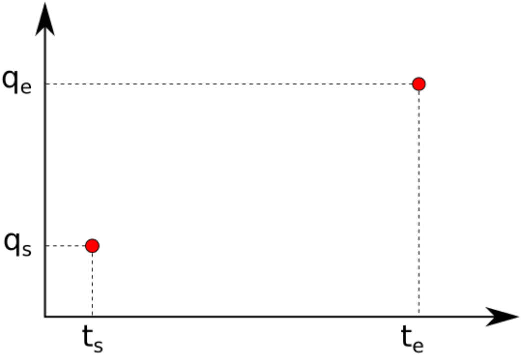


Why trajectories?

- We often do not care only about the final pose of a movement
- It helps in obstacle avoidance
- We can avoid singular configurations

Interpolation basics

Linear interpolation



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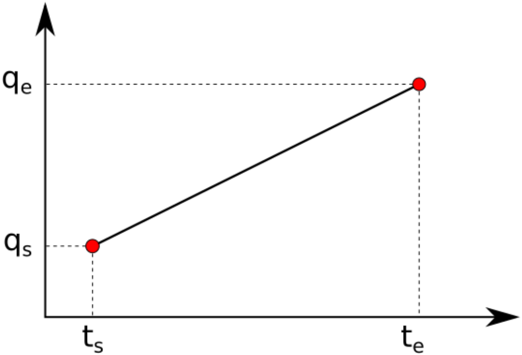

$$q(t) = \left(1 - \frac{t-t_s}{t_e-t_s}\right)q_s + \frac{t-t_s}{t_e-t_s}q_e$$

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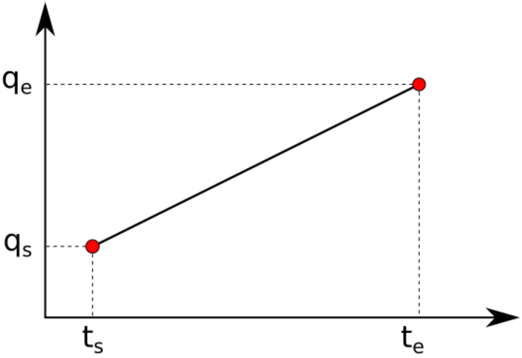

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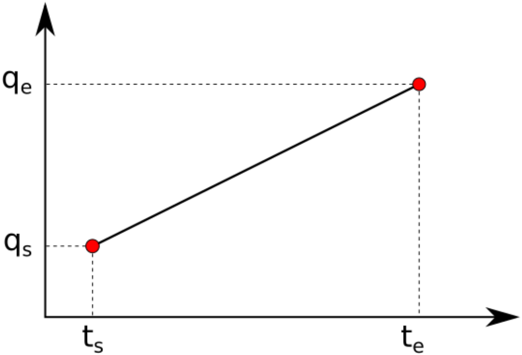

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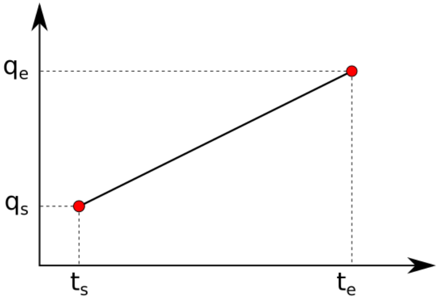

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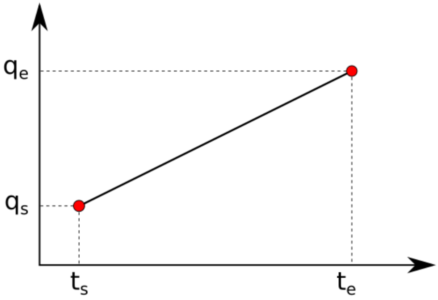

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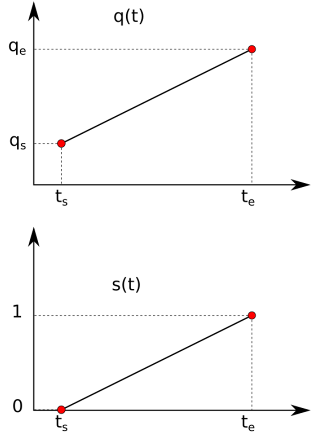

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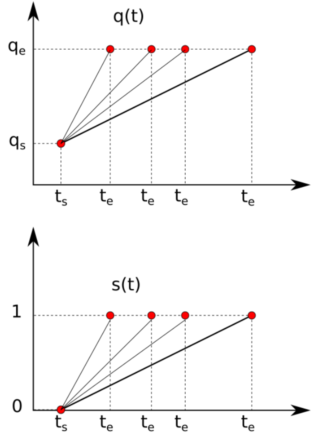

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The 'slope' of s defines the velocity

Interpolation

Application in Robotics

We have initial and target Pose P_s, and P_e, respectively.

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$$q_n(t) = (1 - s(t))q_{ns} + s(t)q_{ne}$$

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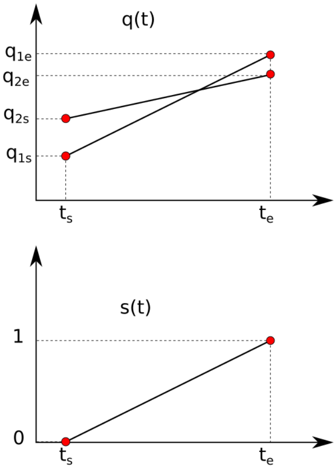

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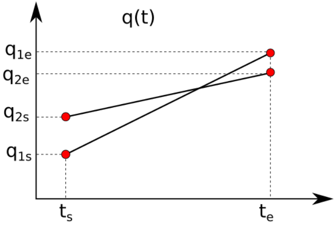
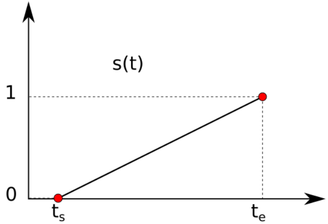
We can either interpolate each joint over the same time period or interpolate each joint to have the same velocity





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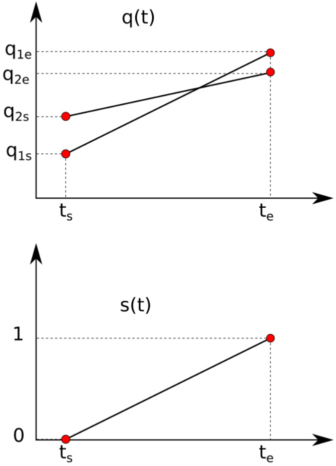
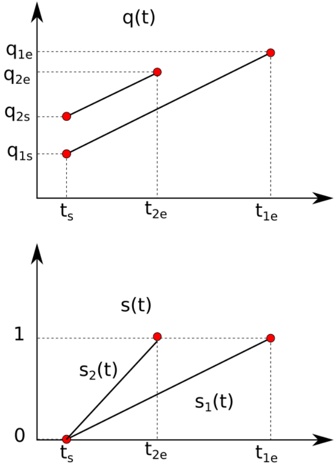
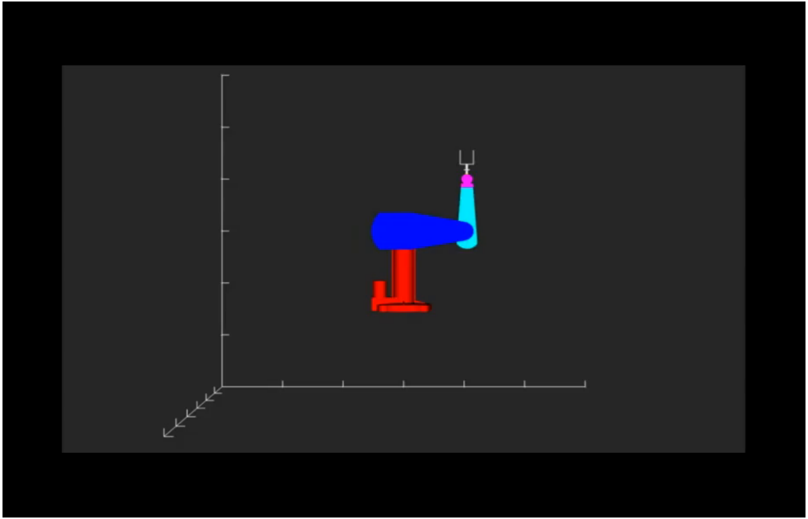




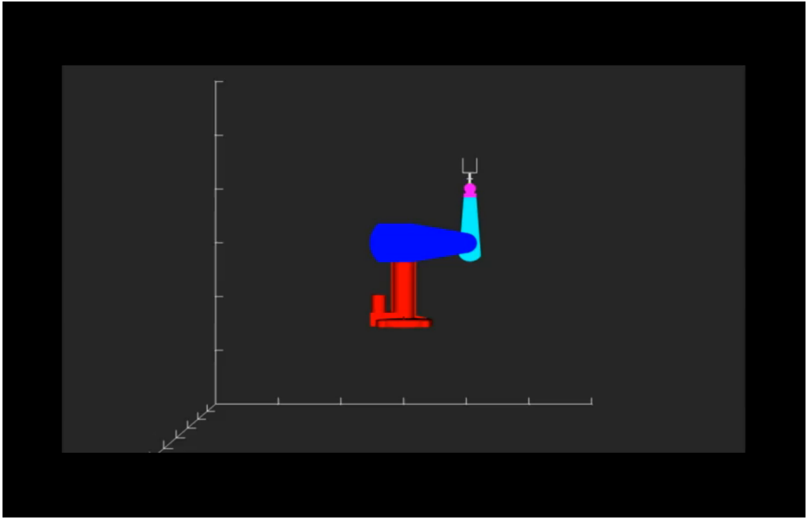
Interpolation

Same duration



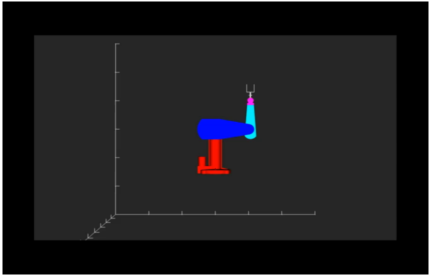
Interpolation

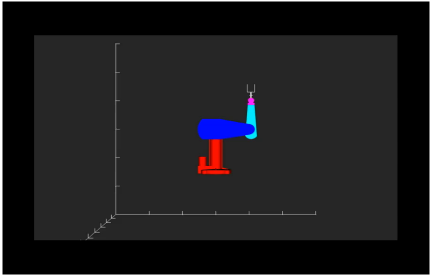
Max velocity



Interpolation

Both





Linear interpolation

Problems?

Are there any issues with the linear interpolation?

Linear interpolation

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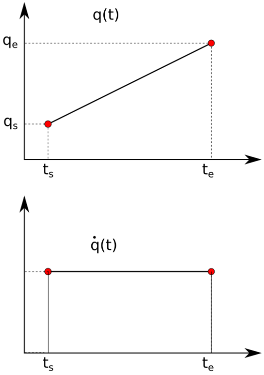

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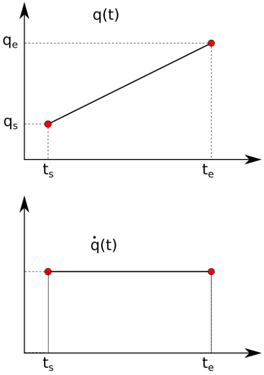

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What about accelerations?

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With conditions:

$$s(0) = 0$$
$$s(T) = 1$$
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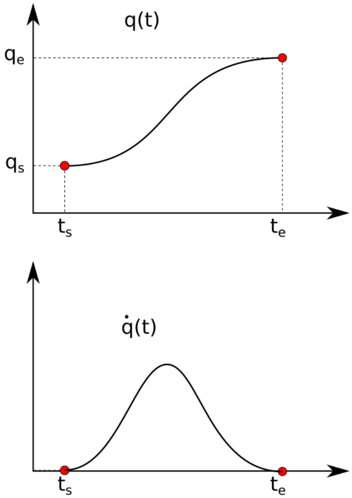
Polynomial Interpolation

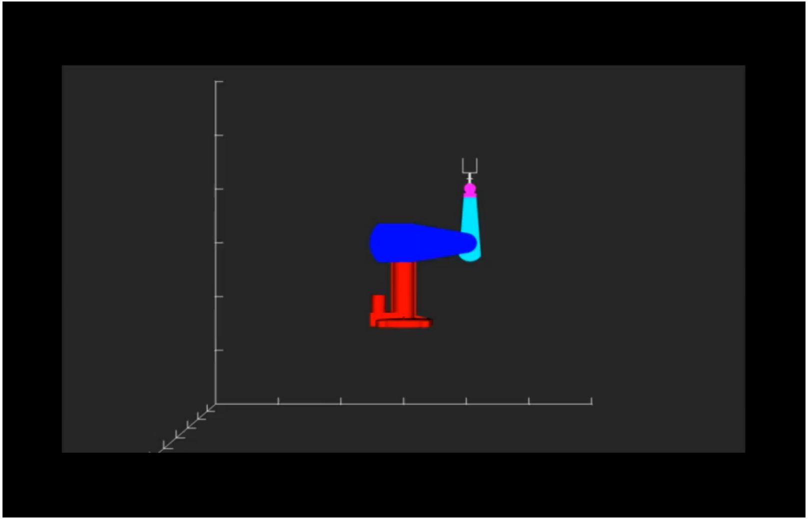
$$\begin{bmatrix} s(0) \\ s(T) \\ \dot{s}(0) \\ \dot{s}(T) \\ \ddot{s}(0) \\ \ddot{s}(T) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ T^5 & T^4 & T^3 & T^2 & T & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix}$$

Polynomial Interpolation



Much smoother movement!

Polynomial Interpolation



What if I want to follow a specific trajectory?

Trajectories

Position

When we only care about the position of the end-effector, things are easy:

Trajectories

Position

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I interpolate each coordinate between starting and ending position (either linear or polynomial interpolation):

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\begin{gathered}
P_x(0) \rightarrow P_x(T) \\
P_y(0) \rightarrow P_y(T) \\
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\text{e.g. } P_x(t) = (1 - s(t))P_x(0) + sP_x(T)
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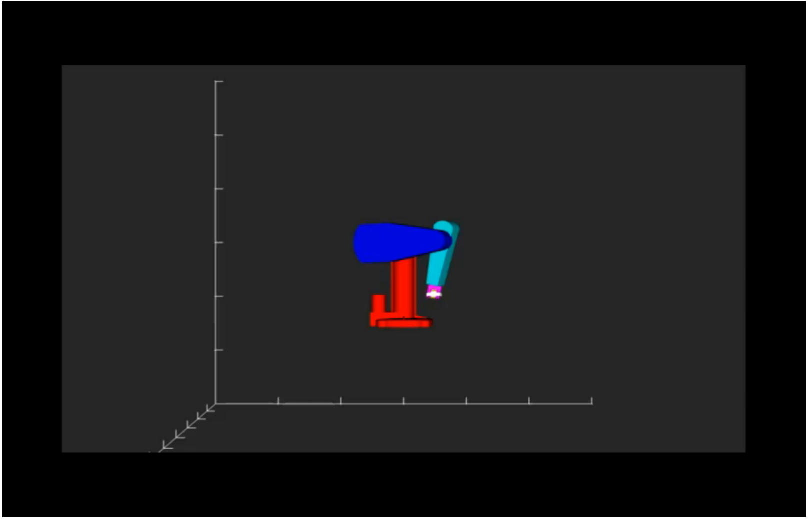
I solve the inverse kinematics for each of the interpolated positions:

$$q(t) = g(P_x(t), P_y(t), P_z(t))$$

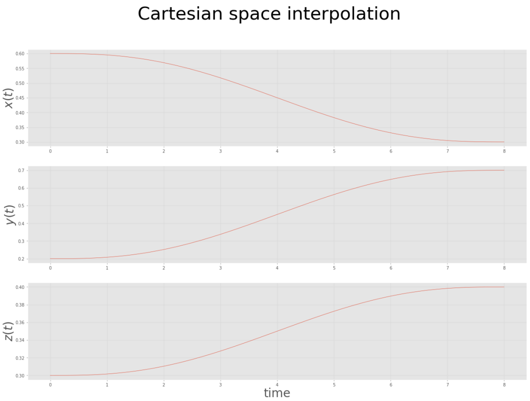
Position Trajectory

With polynomial interpolation



Position Trajectory

Cartesian space and joint space



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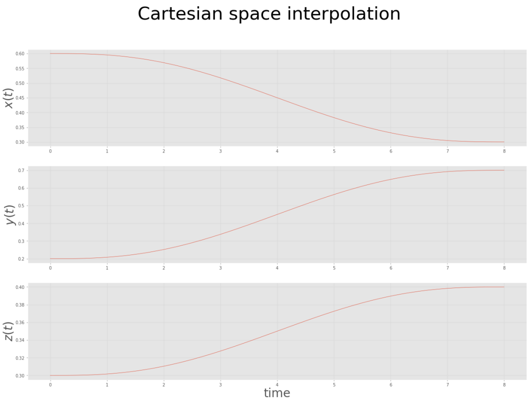

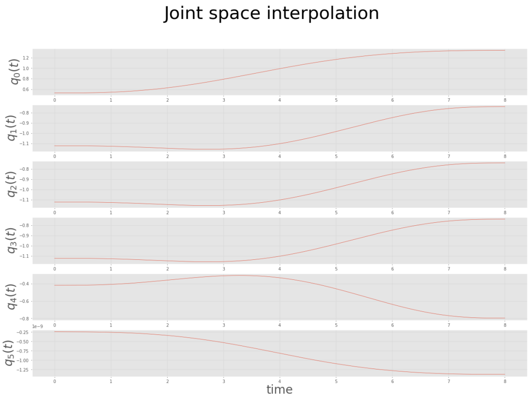

Orientation Trajectory

What about orientation?

Orientation Trajectory

What about orientation? How can we interpolate?

$$R_m^n = \begin{bmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{bmatrix}$$

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We cannot interpolate each value individually :(
We can decompose to Euler angles!

Orientation Trajectories

Euler angles

Arbitrary **orientation**

The transition between two arbitrarily **oriented** coordinate frames can **always** be described in terms of **three elementary rotations**

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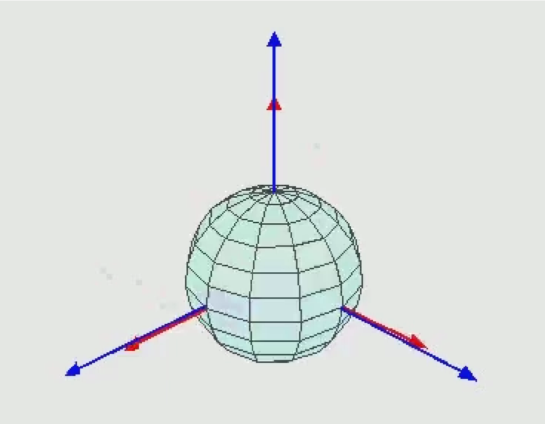

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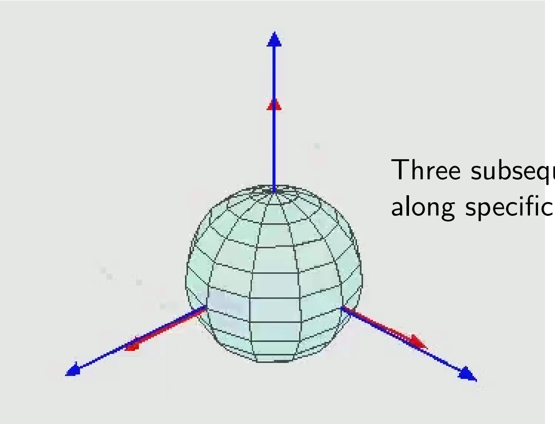

Three subsequent rotations, along specific axes

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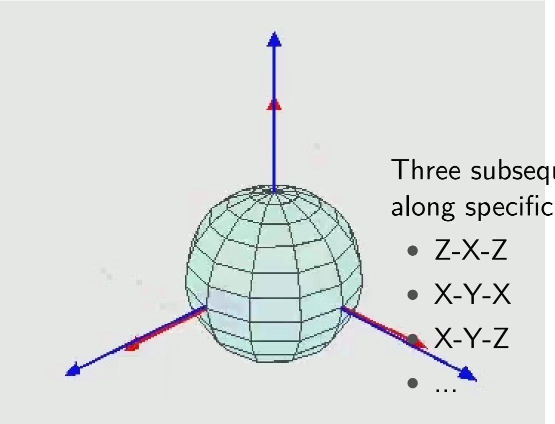

Three subsequent rotations, along specific axes

- Z-X-Z
- X-Y-X
- X-Y-Z
- ...

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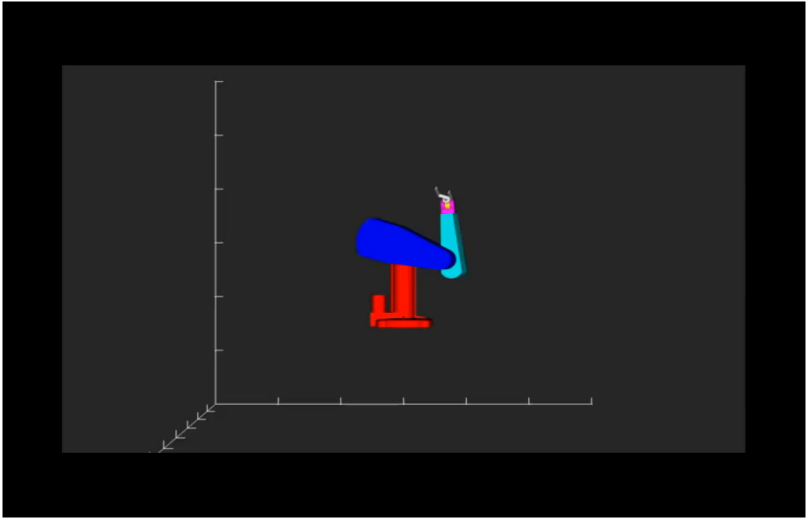
I solve the inverse kinematics for each of the interpolated orientations (and positions):

$$q(t) = g(R(t), P(t), Y(t), P_x(t), P_y(t), P_z(t))$$

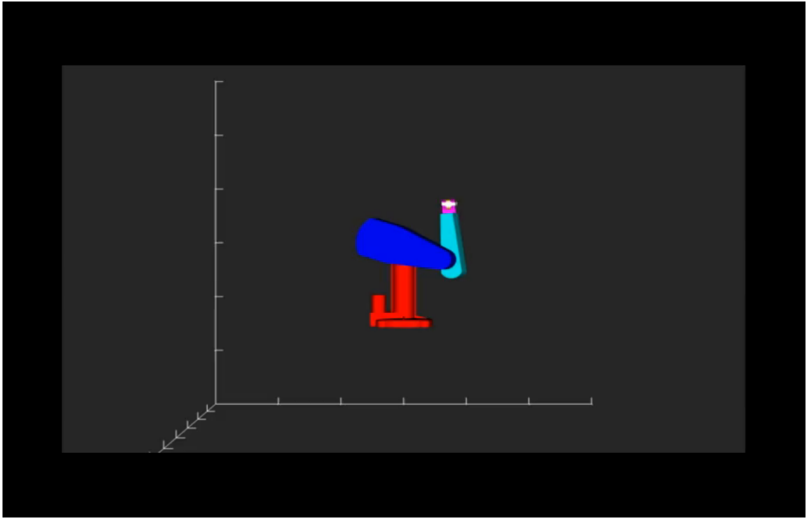
Orientation Trajectory

With polynomial interpolation



Full Trajectory

With polynomial interpolation



Interpolating orientation

Quaternions

Quaternions

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$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

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x	**1**	**i**	**j**	**k**
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

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We can use **SLERP** (Spherical Linear Interpolation)!

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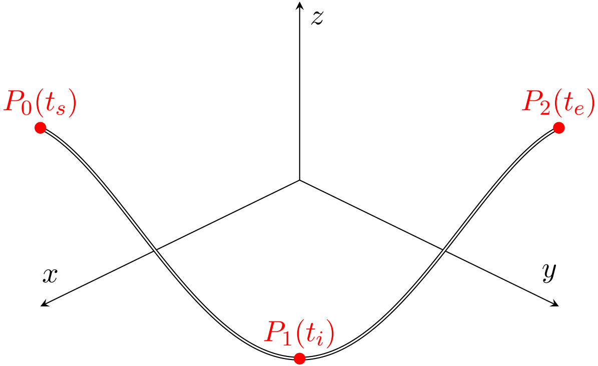
For quaternions it is a bit different:

$$Slerp(q_s, q_e, s(t)) = q_s(q_s^{-1} q_e)^{s(t)}$$

More complex trajectories

Via points

What if we have multiple poses that we want to pass from?



Via points

Higher order polynomials

With conditions:

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Potential problems?

Via points

Multiple polynomials

First polynomial:

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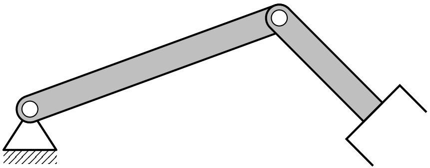
\dot{q}(t_s) &= 0 \\
\dot{q}(t_e) &= \dot{q}_i \\
\ddot{q}(t_s) &= 0 \\
\ddot{q}(t_e) &= 0
\end{aligned}
$$

Second polynomial:

$$
\begin{aligned}
q(t_s) &= q_i \\
q(t_e) &= q_e \\
\dot{q}(t_s) &= \dot{q}_i \\
\dot{q}(t_e) &= 0 \\
\ddot{q}(t_s) &= 0 \\
\ddot{q}(t_e) &= 0
\end{aligned}
$$

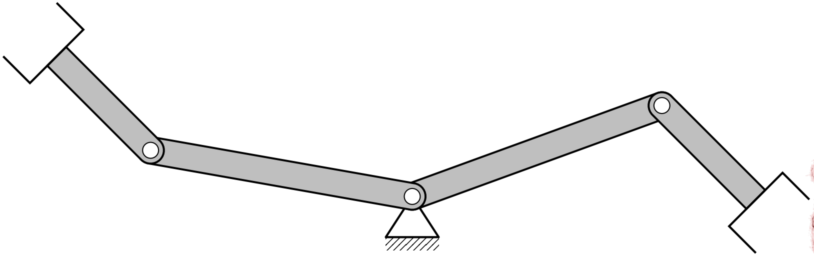
Cartesian path problems

Unrechable poses



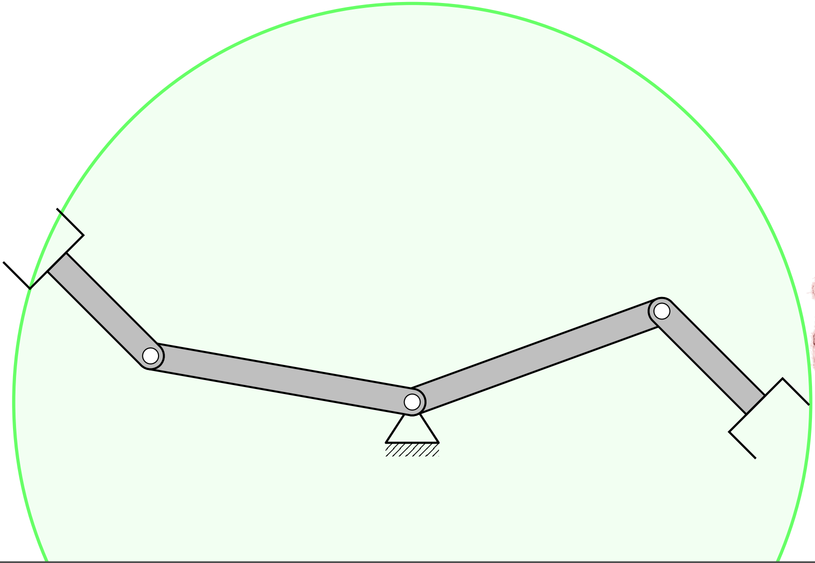
Cartesian path problems

Unrechable poses



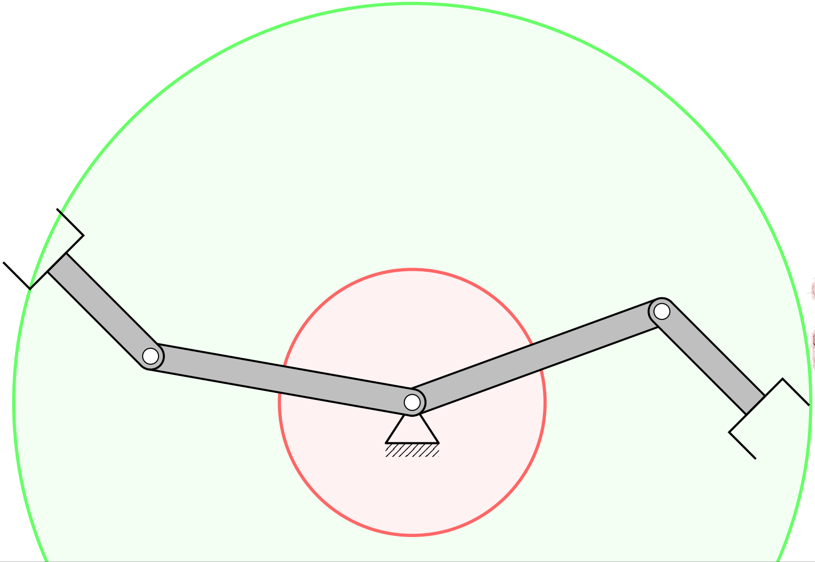
Cartesian path problems

Unrechable poses



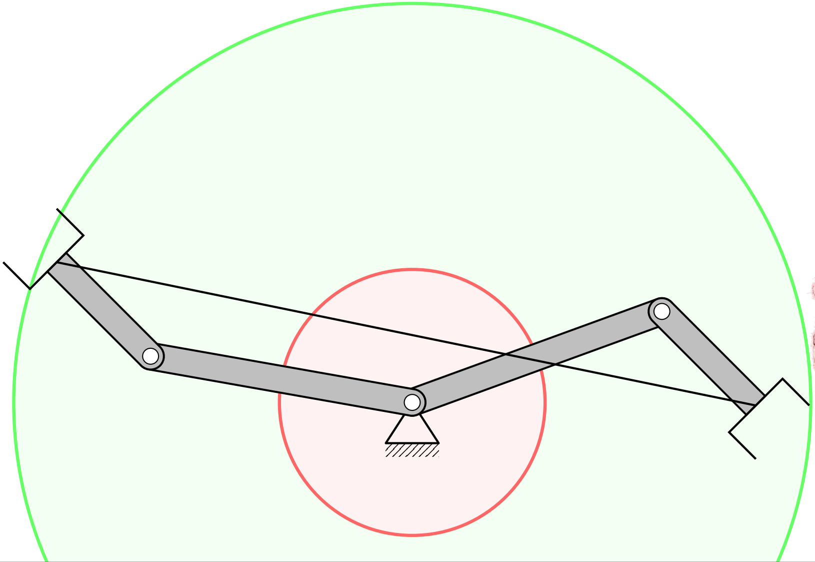
Cartesian path problems

Unrechable poses



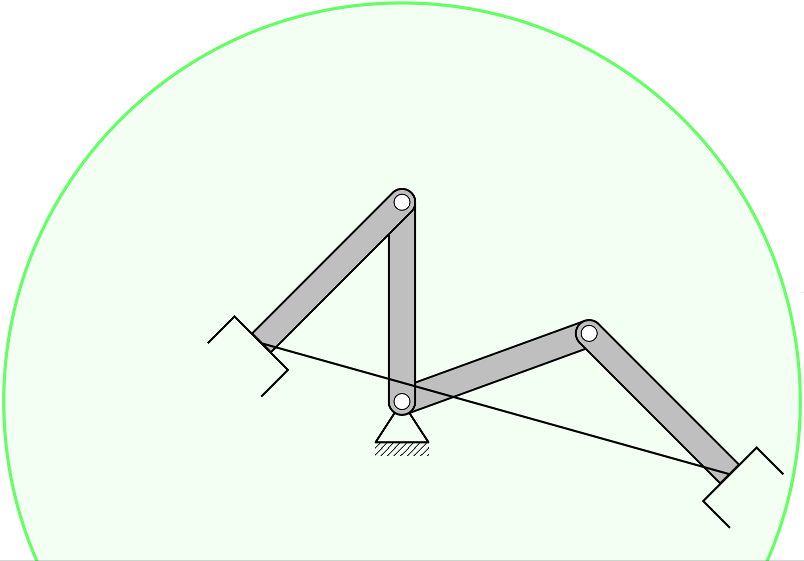
Cartesian path problems

Unrechable poses



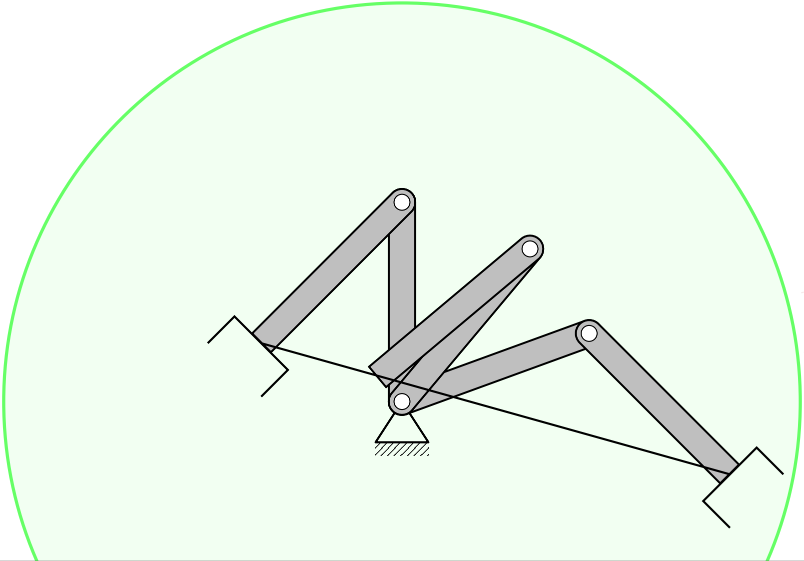
Cartesian path problems

High velocities at singularities



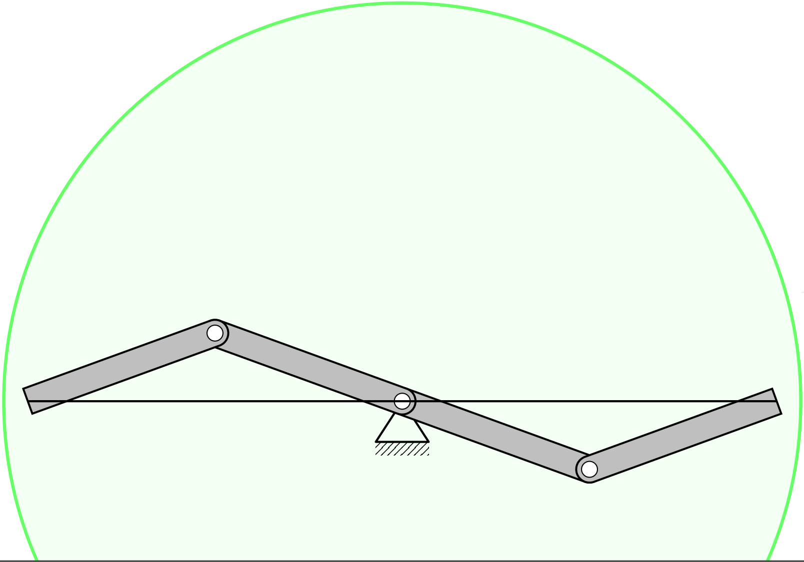
Cartesian path problems

High velocities at singularities



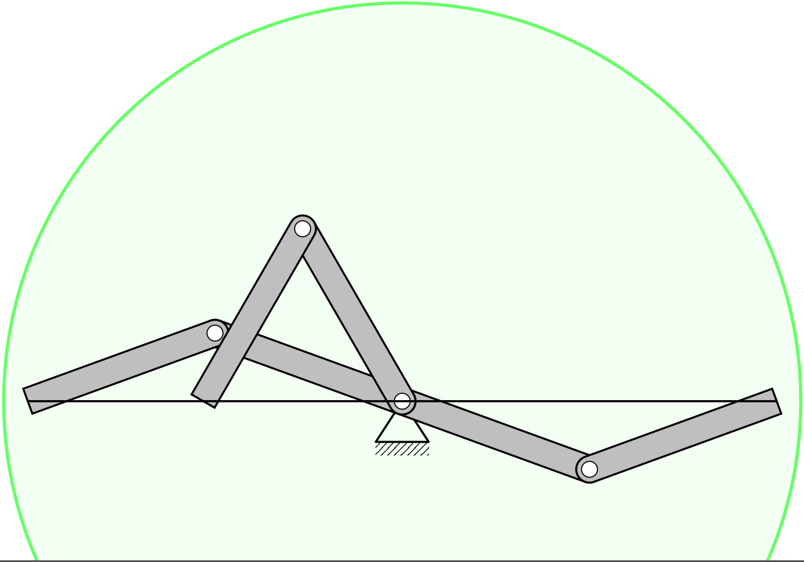
Cartesian path problems

Jumps due to joint limits



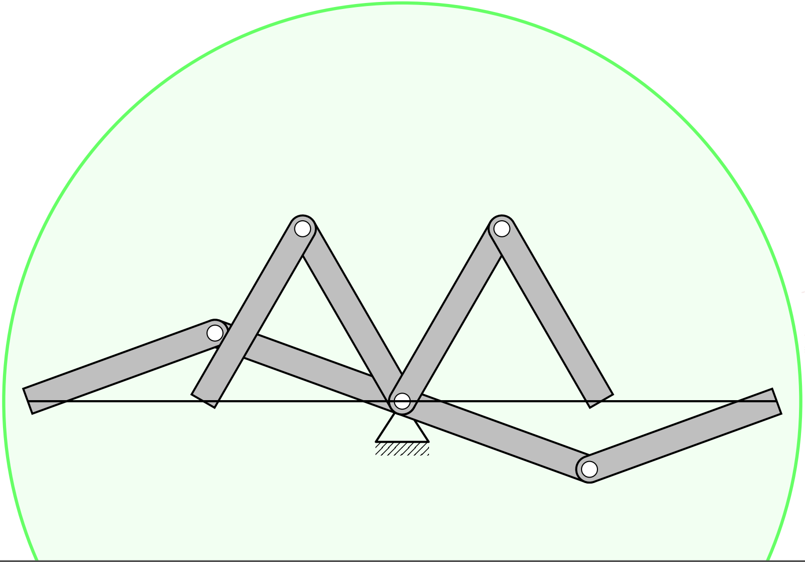
Cartesian path problems

Jumps due to joint limits



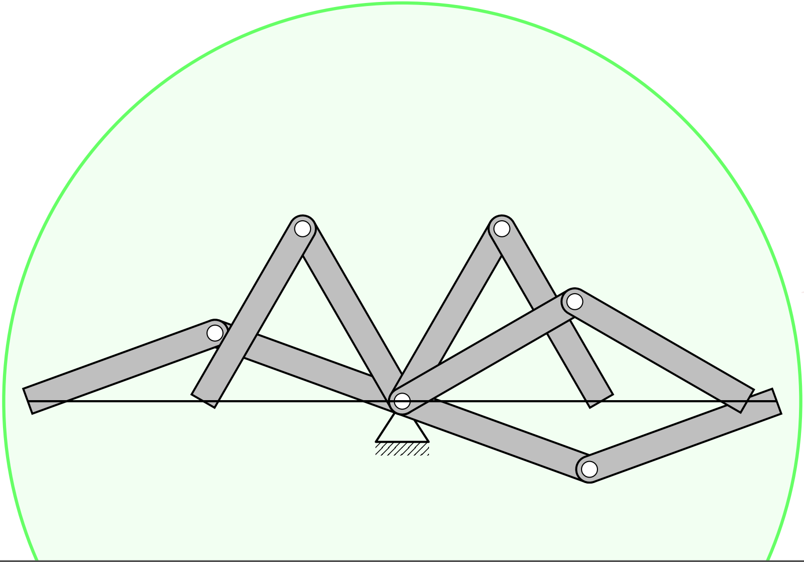
Cartesian path problems

Jumps due to joint limits



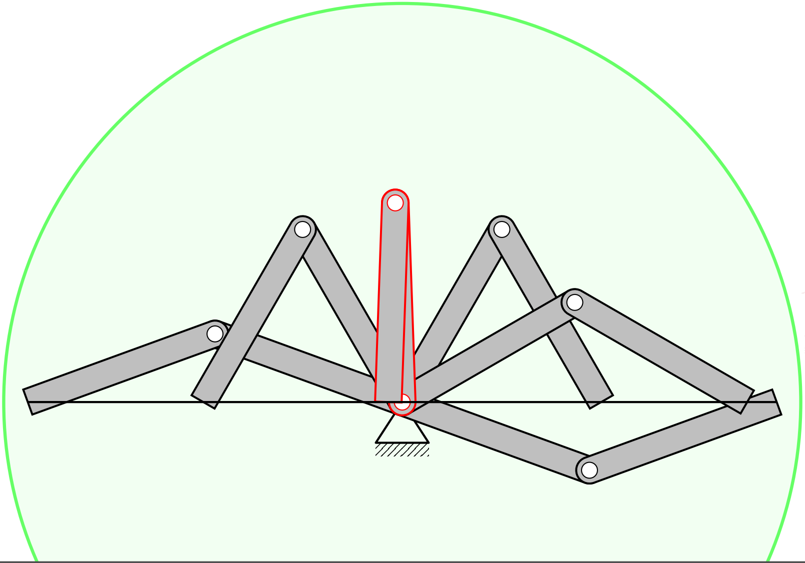
Cartesian path problems

Jumps due to joint limits



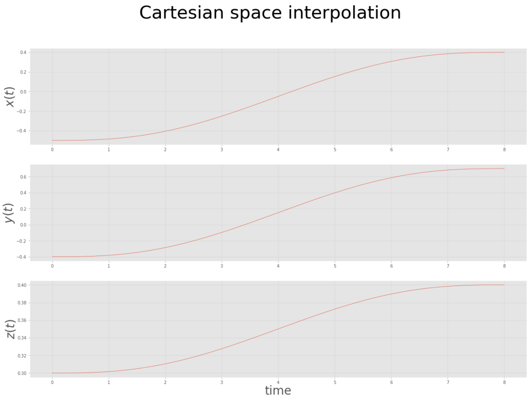
Cartesian path problems

Jumps due to joint limits



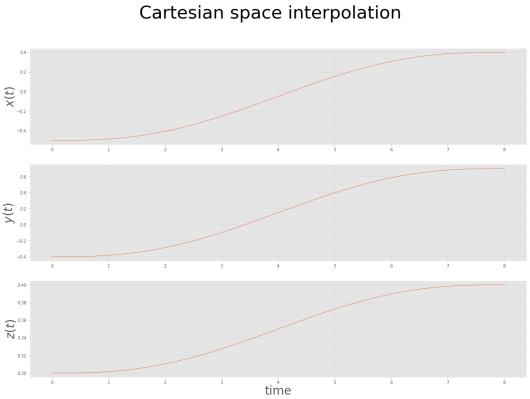
Cartesian path problems

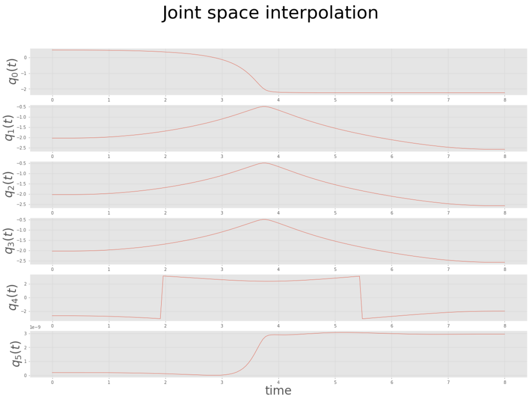
Jumps due to joint limits



Cartesian path problems

Jumps due to joint limits

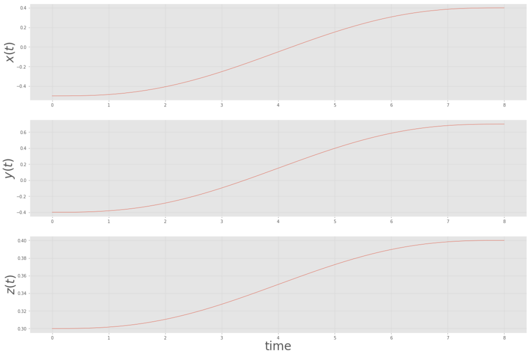




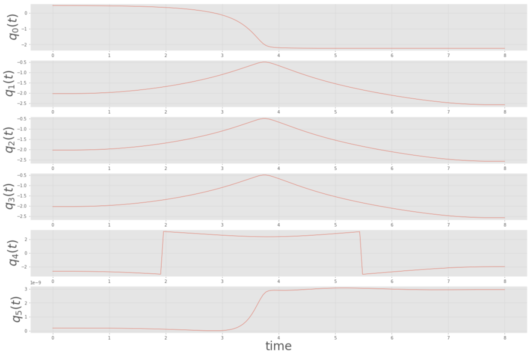
Cartesian path problems

Jumps due to joint limits

Cartesian space interpolation



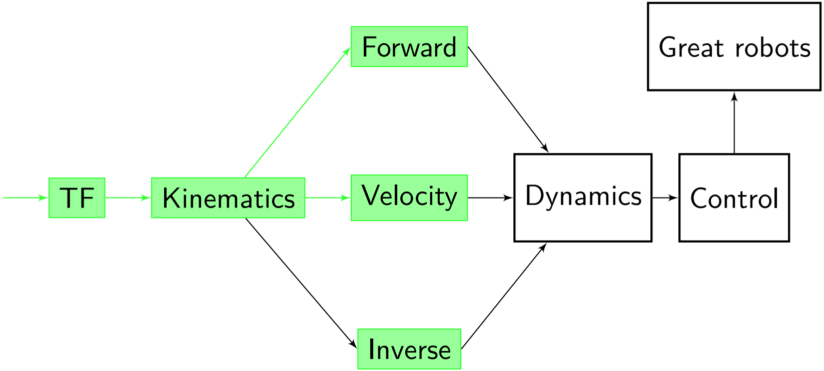
Joint space interpolation



In general, we interpolate in joint space, cause it is more robust

Grand scheme

The big picture



Questions?